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van den Broek, W.A.

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Uncertainty in Differential Games

Bram van den Broek

Preface

Approximately five years ago Willem Haemers told me during a game of badminton that one of his colleagues was looking for a student who was interested in a PhD project in the area of differential games at Tilburg University. This collegue was Jacob Engwerda and one year later I saw an advertisement for the same project. The vacancy was still open and I decided to move to Tilburg. Four years later the project has come to an end and the result is this thesis. I would like to take the opportunity to thank some people who played an important role in the realization of this result.

Daniëlla, the last four years would have been much more difficult without your everlasting support. The moments when I was fully dedicated to my research were not easy for you. In these periods you have been very patient with my absent mind. You are the only person who really understands what I went through to make this project successful. Thank you for your patience and support.

Jacob, without your vision on differential games this thesis would not exist. This vision initiated most of the research. Thank you for having your door always open to discuss many problems and for your guidance through the challenging world of differential games.

Hans, your busy schedule was never an impediment for organizing fruitful meetings. There was always some room in your agenda to discuss one of my problems. During our discussions you always gave the impression of being very relaxed and you were able to give a resfreshing view on my research. Our discussions were often full of mathematical elegance, from which I learned a lot. Thank you for your time and for teaching me how to use simplicity to solve complicated problems.

I thank prof.dr. G. Jank, prof.dr. G.J. Olsder, prof.dr. J.E.J. Plasmans, prof.dr. A.J. de Zeeuw, and dr. P.M. Kort for joining the committee. I am grateful to Tilburg University for making it possible to visit several conferences. At the Eighth International Symposium on Dynamic Games and Applications in The Netherlands in July 1998, I met Gábor Kun from the RWTH Aachen. Afterwards several meetings were organized in Tilburg and Aachen where we had the chance to discuss many interesting problems. In particular the week I spent at the RWTH Aachen in May 2000 has been very fruitful. I enjoyed it a lot. The visit to the Ninth International Symposium on Dynamic Games and Applications in Australia in December 2000 clarified my mind about the finishing touch of this thesis.

During the last four years there was more than work. The social activities around conferences were very enjoyable. They ranged from exploring the nightlife of Groningen by watching soccer matches to bushwalking and canyoning in the Blue Mountains. The weekly indoor soccer matches on Monday were a pleasant start of the week. I enjoyed organizing them. The never ending discussions with Enrico about the new economy and our vision on the world around us were stimulating breaks during the day. The weekends with my friends, family, and many climbing weekends gave me a lot of energy, although sometimes I needed a full Monday to recover. All of these weekends have been a small holiday. Many things will change in the future but this will always remain the same.

Bram, April 2001.

PROMOTOR	:	prof.dr. J.M. Schumacher
COPROMOTOR	:	dr. J.C. Engwerda

Uncertainty in Differential Games

PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Katholieke Universiteit Brabant, op gezag van de rector magnificus, prof.dr. F.A. van der Duyn Schouten, in het openbaar te verdedigen ten overstaan van een door het college voor promoties aangewezen commissie in de aula van de Universiteit op vrijdag 15 juni 2001 om 14.15 uur door

WILHELMUS ABRAHAM VAN DEN BROEK

geboren op 19 augustus 1972 te Etten-Leur.

Contents

Chapter 1

Introduction

1.1 Dynamic Game Theory

Game theory is concerned with the modeling of situations in which several players[1] have to make a decision and each possible combination of decisions leads to a different result, which is valued by each player in his own way. Given a model of such a situation, the theory studies several possible outcomes of the game and investigates which outcomes satisfy desirable properties for the players. If the players act *cooperatively*, one usually looks for a combination of decisions leading to a result in which no player can improve his value without worsening the value of at least one other player. Such outcomes are called *Pareto optimal* solutions. In contrast to a cooperative game, each player can also optimize his own value without taking into account the objectives of the other players, i.e. the players act *noncooperatively* in this case. In a noncooperative game, one usually looks for a combination of decisions that leads to a result in which no player can improve his value, given the decisions of the other players, by deviating unilaterally. Such an outcome is called a *Nash equilibrium* solution. The following example of a game with two players has a Pareto optimal and a Nash equilibrium solution. Due to its simplicity it clearly shows the features of cooperation and noncooperation.

Example 1.1.1 Probably the most famous example of a game is the *prisoner's dilemma*. This example can be found in any book on game theory; see for instance [89, p. 236] or [10, p. 83]. The story behind this game is as follows. Two criminals are suspected of having committed a serious crime for which they are arrested. Since there is no direct evidence against them, their

[1] Synonyms are "agents", "persons", "decision makers", "parties", or "authorities".

conviction depends on whether they confess or not. They are put in separate cells and interrogated privately. If they both confess the crime, they will be sentenced to 5 years. If only one of them confesses and puts the blame on the other one, he will be sentenced to 1 year and the other will be sentenced to 10 years. However, if neither one confesses, they will be convicted of a lesser crime and sentenced to 2 years. The situation is illustrated in Figure 1.1. The numbers in the matrix are the number of years the prisoners are sentenced. The first number belongs to prisoner 1 and the second number belongs to prisoner 2.

		prisoner 2	
		confess	don't confess
prisoner 1	confess	5,5	1,10
	don't confess	10,1	2,2

Figure 1.1: Prisoner's dilemma.

Clearly, the best joint solution for both prisoners is not to confess. Stated differently, the combination of decisions "don't confess, don't confess" is a Pareto optimal solution. However, this cooperative solution is highly unstable, since each player has an incentive to deviate to the option of confessing. If they both deviate, i.e. if they both confess, they end up in a worse solution than the Pareto optimal solution. Nevertheless, the combination of decisions "confess, confess" is a Nash equilibrium solution: no player has an incentive to deviate unilaterally. □

1.1.1 Dynamics and Information

In the prisoner's dilemma game the players make their decisions simultaneously. Alternatively, the rules of the game could be such that prisoner 1 first announces his decision, and prisoner 2 is able to observe this decision and uses this information to make his decision. If prisoner 1 chooses to confess, the best choice for prisoner 2 is also to confess. If prisoner 1 chooses not to confess, again the best choice for prisoner 2 is to confess. Prisoner 1 is aware of this strategic behavior of prisoner 2, so he chooses to confess. Subsequently, prisoner 2 confesses. These rules of the game are probably most interesting for the police force, since the prisoners are enforced to confess! A game in which one player announces his decision first and determines the decision of the other player is called a *Stackelberg game*. The player who decides first is called the *leader* and the player who decides next is called the *follower*.

Instead of decisions, one usually distinguishes between *actions* and *strategies* [10]. If an expedition team has to decide whether or not to climb the Mount Everest on some day, then a strategy is: "if the weather forecast for that day predicts perfect climbing conditions, then we start the climb, otherwise we stay in camp 4". What the climbers will do on that day, depends on the weather conditions, which is something not known beforehand by the expedition leader. The alternative actions are to go climbing or to stay in camp 4. In the Stackelberg prisoner's dilemma game as discussed above the strategy for prisoner 2 is: "If prisoner 1 confesses, I confess; if prisoner 1 does not confess, I will". Once prisoner 1 has announced his action to confess, the action of prisoner 2 is to confess too.

The following abstract example illustrates the role of the information in a game and how it affects the solution. Similar examples can be found in [10, Example 1.2] and [110].

Example 1.1.2 Consider the situation where two firms are competing on the same market during two periods of time. During each period the firms have to make strategical decisions (e.g. advertising, investments) that influence the profits in a certain way. The objective of the firms is to maximize their total profits over both periods. The situation and the possible expected profits are shown in Figure 1.2. The profits are measured in euros. Clearly the values are not realistic; it is the ratio between these values that matters. The decision points in the first and the second period

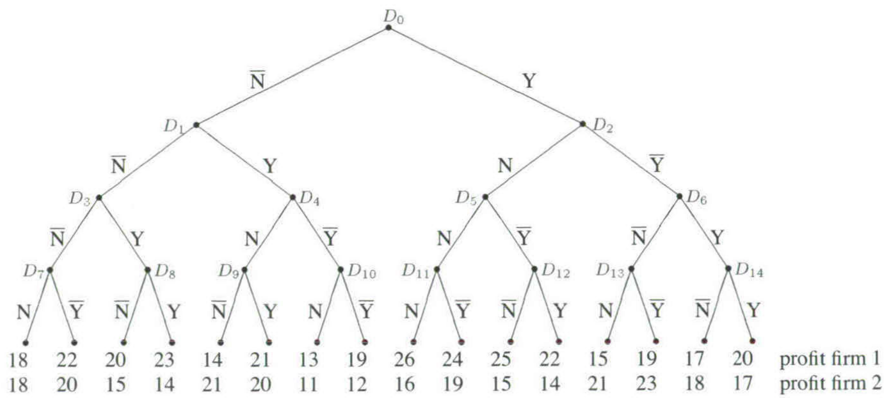


Figure 1.2: Two-period game between two firms.

for firm 1 are D_0 and $D_3, \ldots, D_6$, respectively. The decision points in the first and the second period for firm 2 are D_1, D_2 and $D_7, \ldots, D_{14}$, respectively. At each decision point a firm has two possible actions denoted by the symbols N and Y. The game has 16 possible outcomes. Assume that the firms are not cooperating. For three different types of information structures, Nash equilibrium solutions can be determined as follows.

(i) The order of the actions will be taken as depicted in Figure 1.2: in each period firm 1 decides first and firm 2 decides second. Firm 2 is able to observe the decision of firm 1 in both periods and uses this information to decide between its alternative actions. At the beginning of the second period the firms know to which decision point the game has evolved. The solution of the game under this information structure can be found by working backward in time. From each of the decision points $D_7, \ldots, D_{14}$, firm 2 chooses the action Y or N, depending on which action leads to higher profit. For instance, at decision point D_7, firm 2 chooses for the action Y, since 20 euro is a higher profit than 18 euro. This optimal action is denoted by a bar in Figure 1.2. The optimal actions for the decision points $D_8, \ldots, D_{14}$ are denoted similarly. At decision point D_3, firm 1 knows that if it chooses the action N, firm 2 will choose the action Y subsequently, and if he chooses the action Y, firm 2 will choose the action N. Since 22 euro is a higher profit than 20 euro, firm 1 chooses the action N at decision point D_3. This procedure can be repeated until for the remaining decision points; each optimal action is denoted by a bar in Figure 1.2. Apparently, the Nash equilibrium solution for the information structure under consideration goes along the decision points D_1, D_3, and D_7, and consists of the action N for firm 1 in both periods and the consecutive actions N and Y for firm 2. The resulting profits for firm 1 and firm 2 are 22 euro and 20 euro, respectively.

(ii) The second information structure consists of simultaneous decisions during each period. Furthermore, at the beginning of the second period the firms have observed the actions of the first period. Stated differently, at the beginning of the second period the firms know to which of the decision points $D_3, \ldots, D_6$ the game has evolved. At decision point D_3 the firms have to announce an action simultaneously; each firm has to make a decision without knowing what the other firm will do. The four possible outcomes are depicted in Figure 1.3. The pair of numbers in each box represents the profits for both firms resulting from the corresponding decisions. If firm 1 chooses the action Y and firm 2 chooses the action N, both firms will not have an incentive to deviate unilaterally. Indeed, if firm 1 deviates unilaterally, its profit changes from 20 euro to 18 euro, and if firm 2 deviates unilaterally, its profit changes from 15 euro to 14 euro. Apparently, the combination of actions "Y for firm 1, N for firm 2" is a Nash equilibrium solution for the game depicted in Figure 1.3.

		firm 2 N	firm 2 Y
firm 1	N	18,18	22,20
firm 1	Y	20,15	23,14

Figure 1.3: Second-period game starting from decision point D_3.

It is easily verified that this is the only such solution of this game. In a similar way, one can determine unique Nash equilibrium solutions for the second-period games starting from the decision points D_4, D_5, and D_6. The results are shown in Table 1.1. The profit pair

starting point	action firm 1	action firm 2	profit firm 1	profit firm 2
D_3	Y	N	20	15
D_4	N	N	14	21
D_5	N	Y	24	19
D_6	Y	N	17	18

Table 1.1: Nash equilibrium solutions for the second-period games.

corresponding to these solutions can be attached to the decision points $D_3, \dots, D_6$, which results in the first-period game depicted in Figure 1.4. It is easily verified that the Nash equilibrium solution of this game is the combination of actions "Y for firm 1, N for firm 2", leading to the profits: 24 euro for firm 1 and 19 euro for firm 2. Summarizing, with the information structure under consideration, the Nash equilibrium solution for the two-period game consists of the consecutive actions Y and N for firm 1 and the consecutive actions N and Y for firm 2. This is clearly a different solution than the one obtained under the information structure in case (i). It is interesting to note that firm 1, having a more profitable information structure in case (i) (firm 1 acts as a leader in the Stackleberg-sense), has a higher profit in the Nash equilibrium solution under the symmetric information structure considered in this case. This shows that a more profitable information structure does not necessarily lead to a higher profit.

(iii) The third information structure consists of announcing the actions for both periods simul-

		firm 2: N	firm 2: Y
firm 1	N	20,15	14,21
	Y	24,19	17,18

Figure 1.4: First-period game under consecutive simultaneous decision rules.

taneously at the beginning of the first period. The firms have four possible actions: NN, NY, YN, and YY, where the first and the second symbol refer to the action in the first and the second period, respectively. There are 16 possible outcomes, which are shown in Table 1.2; the actions that are ordered vertically (horizontally) correspond to firm 1 (firm 2). It is

	NN	NY	YN	YY
NN	18,18	22,20	14,21	21,20
NY	20,15	23,14	13,11	19,22
YN	26,16	24,19	15,21	19,23
YY	25,15	22,14	17,18	20,17

Table 1.2: Two-period game under the third information structure.

easily verified that the combination of actions "YY for firm 1, YN for firm 2" is the Nash equilibrium solution of this game, leading to the profits: 17 euro for firm 1 and 18 euro for firm 2. Note that both firms have a higher profit in the Nash equilibrium solution under the information structure in case (ii); in that structure the firms have more information available. Thus in this example more information for both firms leads to higher profits.

□

This example illustrates that different information structures require different techniques to determine Nash equilibrium solutions, and in general, one obtains different solutions. The information structure considered in the second case is usually referred to as a *feedback* information structure. The firms are aware of the situation at the beginning of the second period and use this updated information to make a decision for the second period. The information structure considered in the third case is called an *open-loop* information structure. The firms make binding agreements at the beginning of the first period about the actions they undertake during both periods, i.e. in an open-loop Nash equilibrium the period of commitment equals the entire planning period.

A game in which a number of decisions have to made through time is called *dynamic*. The two-period game of Example 1.1.2 is an example of a dynamic game. A typical aspect in a dynamic game is the evolution of the "state" of the game. The state of the two-period game could for instance be the number of customers the firms have. This number evolves from an initial state to certain states after each period. The state is measured at discrete points in time. A game with such a state evolution is called a *difference game*. Games of this type with a finite set of alternative decisions have been extensively studied in [10].

Rather than stepwise as in difference games, the state of a dynamic game can also evolve continuously through time. Such a game is called a *differential game*. As an example, consider the following continuous-time version of an advertising model, which has been presented in [97]. In this game, two firms can advertise with a certain intensity $u_1(t)$ and $u_2(t)$ (amount of money spent on advertising per unit of time) during a fixed period: $0 \leq t \leq T$. These intensities are normalized and bounded: $0 \leq u_i(t) \leq 1$. The total number of customers is constant and normalized to 1. At time t, the fraction of customers of firm 1 (the state) and firm 2 are denoted by $x(t)$ and $1 - x(t)$, respectively. If firm 1 advertises, its fraction of customers increases proportionally to its advertising intensity and the fraction of customers of firm 2. Similarly, if firm 2 advertises, the fraction of customers of firm 1 decreases proportionally to firm 2's advertising intensity and the fraction of customers of firm 1. This state dynamics can be described by the differential equation

$$\dot{x}(t) = (1 - x(t))u_1(t) - x(t)u_2(t).$$

Here, $\dot{x}$ denotes the time derivative of x. Per unit of time, each firm's profit is equal to a certain positive constant c_i times its fraction of customers, minus his advertising intensity. The total profit equals this quantity integrated over the time horizon from 0 to T. The objective of each firm is to maximize these profits by choosing an appropriate advertising intensity, i.e. the criteria of firm 1 and firm 2 are

$$\max_{u_1} \int_0^T (c_1 x(t) - u_1(t))dt \quad \text{and} \quad \max_{u_2} \int_0^T (c_2(1 - x(t)) - u_2(t))dt$$

respectively. Also in this continuous-time model, the information structure plays an important role. Suppose that the firms are aware of the initial customer allocation x_0 and that they are not allowed to use information about customer allocations at other points in time; they may even be unable to observe future allocations until the end of the period. Using the initial information, the firms choose advertising functions u_i, thereby announcing their advertising intensity for the whole period from time 0 to time T. If the firms are limited to such a choice, we say that their information structure is an *open-loop pattern*. Alternatively, if both firms are able to observe the customer allocation at any point in time, and to adapt their advertising intensities accordingly, we say that their information structure is a *feedback pattern*. In this case, the firms choose feedback strategies instead of control functions (actions) in the open-loop case. Although this advertising model seems rather simple, analyzing it is a different story; especially under the feedback information structure. Due to the constraints on the advertising intensities, one ends up with value functions with singular surfaces of which little is known in this context [13, 96].

In general terms, a two-player differential game involves a differential equation

$$\dot{x} = f(t, x, u_1, u_2), \quad x(0) = x_0$$

and for each player a criterion, which consists of an integral and a final time-penalty term:

$$J_i(u_1, u_2) = \int_0^T g_i(t, x, u_1, u_2)dt + q_i(x(T)).$$

If player 1 aims to minimize $J = J_1$ and player 2 aims to maximize the same criterion, the game is said to be a *zero-sum* differential game. This terminology is justified by the fact that player 2 aims to minimize the criterion $J_2 = -J$, so that the sum of the criteria, i.e. $J_1 + J_2$, is zero. If the game is not a zero-sum differential game it is said to be a *nonzero-sum* differential game.

Two well-established applications of zero-sum dynamic game theory are *pursuit-evasion games* [10, Chapter 8] and *worst-case design* of controllers; for instance H_∞ control theory can conveniently be embedded in a zero-sum dynamic game context [9]. In pursuit-evasion games the final time T is endogenously determined; for instance $T = \inf\{t | \alpha(t, x(t)) = 0\}$ for some scalar function α. In contrast to that, in economic applications one usually fixes the final time or, alternatively, one takes it to be infinite (a case in which it is usually additionally assumed that $q_i = 0$). Also in engineering applications one distinguishes between finite and infinite-horizon differential games; for instance, the time-domain interpretation of an H_∞ norm can be finite or infinite.

Several information structures can be considered in a differential game. Most commonly used are the *open-loop* and the *feedback* information patterns. In the open-loop case the players only have access to the initial state, and in the feedback case they have access to the state at time t. Other structures are for instance (i) the *closed-loop perfect state pattern* which consists at time t of the complete state history from time 0 to time t, and (ii) the *memoryless perfect state pattern* which consists at time t of the initial state and the state at time t. Under any of these information structures a Nash equilibrium[2] is defined as a pair of strategies (u_1^*, u_2^*) such that no player has an incentive to deviate unilaterally, i.e. (assuming that the players aim to minimize the integral criteria) if $J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*)$ for all u_1 and $J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2)$ for all u_2. Pontryagin's minimum principle [101] can be applied to obtain a set of necessary conditions which are satisfied for open-loop Nash equilibria. Dynamic programming [17] is usually applied to study feedback Nash equilibria. Open-loop Nash equilibria are less realistic. However they are easier to study; they often lead to analytic tractability. In contrast, feedback Nash equilibria are closer to reality, but much more difficult to analyze; numerical techniques are often the only tool to study them.

Due to their simple structure, *linear-quadratic* (LQ) games are of special interest. This structure consists of state dynamics described by a linear differential equation and quadratic integral criteria. Since LQ control theory was already well developed in the beginning of the 70s, it was expected in the 70s and the 80s that the same level of maturity could be reached for LQ differential game theory. In connection with to the successful development of H_∞ control theory, this level of maturity has indeed been reached for zero-sum LQ differential games. However, in a nonzero-sum context, there are still many unsolved and challenging problems. Although a survey paper [60] and some mathematically rigorous papers [82, 83] followed shortly after the papers [113, 114] which initiated nonzero-sum LQ game theory, one can observe during the last three decades several misunderstandings concerning existence and uniqueness questions. For instance, some authors claimed that in the finite-horizon open-loop case a unique Nash equilibrium always exists and moreover that it can be obtained from a set of asymmetric coupled Riccati differential equations. However, Eisele [43] presented several examples showing that open-loop Nash equilibria need not to be unique. Engwerda [45] presented further results in this direction. He showed that the set of asymmetric Riccati differential equations may have no solution, even when an open-loop Nash equilibrium exists. He also showed that the finite-horizon game may have a unique open-loop Nash equilibrium for each horizon length, while the corresponding infinite-horizon game has multiple open-loop Nash equilibria. Furthermore, the limit of the open-loop Nash equilibrium obtained by letting the final time approach infinity, is not necessarily an open-loop Nash equilibrium for the infinite-horizon game[3]. The limiting behavior of the solutions of the open-loop Riccati differential equations has also been investigated in [4]; in a discrete time context, this limiting behavior of open-loop Nash and Stackelberg Riccati difference equations has been addressed in [52]. In the feedback case, it has been claimed in [79] that the algebraic Riccati equations corresponding to the infinite-horizon feedback LQ differential game have a unique pos-

[2] A Nash equilibrium is a solution concept for a noncooperative game. Other solution concepts exist for cooperative games. See [58] for a review on cooperative differential games.

[3] A similar result has been obtained under the feedback information structure in [99].

itive semi-definite solution under some weak conditions. However, Weeren et al. [122] showed that already in the scalar case these equations generally have either one or three solutions, thus refuting the claim in [79]. Precise conditions under which one or three solutions exist are given in [46]. Summarizing, both open-loop and feedback LQ differential games have not reached the same level of maturity as LQ control theory. Although widely applied in economic models, a theoretical basis in LQ dynamic game theory is still developing. For instance, global existence results for the feedback Riccati differential equations have recently been obtained in [51] and [122]. The authors of both papers stress the importance of easily verifiable conditions for existence, uniqueness, and convergence of the solutions of these equations. Furthermore, more research on the corresponding algebraic Riccati equations is required in order to obtain numerical solutions.

1.1.2 Applications

Dynamic game theory has many applications in economics and control engineering. In economics one naturally encounters conflict situations. Game theory is an appealing way to model economic conflict situations. Here we shall discuss a few applications which are naturally modeled by dynamic game theory. In control engineering one uses game theory e.g. in worst-case design of controllers. This will be discussed later.

Environmental economical problems can conveniently be modeled using dynamic game theory. Many production processes damage the environment. For instance the greenhouse effect is a result of modern technology. Dasgupta [37] has introduced a control model which analyses the intertemporal trade-off between the benefits of production and the resulting damage to the environment. This damage is caused by pollutants which cross national borders (through air or rivers). Several countries are involved in this pollution process. The work of Dasgupta has been extended by van der Ploeg and de Zeeuw [100] to a dynamic game setting where the countries are the players and where the state of the game is the average stock of pollutants spread out over all countries involved. The countries control the pollution process by emission charges. In [100], it is shown that the feedback Nash equilibrium leads to a higher concentration level of pollutants and lower emission charges than the less realistic open-loop Nash equilibrium. A similar result has been found by Mäler and de Zeeuw [86] in the context of an acid rain differential game. They model the damage to the environment as a buffer stock which is depleted as long as the deposition exceeds a certain critical load. They show that for both the open-loop and the feedback Nash equilibrium the depositions in each country converge to the critical loads in the long run. However, the steady state value of the buffer stock in the feedback Nash equilibrium is lower than in the open-loop Nash equilibrium, which implies higher depletion levels and hence more damage to the environment.

An interesting area in models of environmental problems is the theory around *common-property resources*; see for instance [34] or [35]. Examples of such resources are ground water, fish populations [78], or pastures [16]. The models have in common that a certain number of users (the players) has access to a resource (the state), which is either exhaustible or renewable. The richness of the resource clearly depends on the way it is controlled by the users. Additionally a resource can be polluted when it is used as a waste sink. Mäler et al. [87] introduce a model describing the dynamics of ecological systems such as shallow lakes. A typical element in this example is that shallow lakes flip at a certain point from a clear state to a turbid state. This is due to agricultural activities around the lake and its use as a waste sink. In order to prevent the lake from flipping, coordination is required. The model in [87] is a nonlinear differential game, the open-loop Nash equilibrium of which is computed. It is shown that for a small number of communities loading phosphorus in the lake, a constant tax on this loading leads to a clear lake. The feedback Nash equilibrium is more realistic but due to the nonlinearity it seems to be too difficult to implement it in this particular problem. It is however shown that it is to be expected that this equilibrium results in higher amounts of phosphorus. This is in line with the international pollution models discussed above: pollution is higher in a feedback Nash equilibrium and since this concept is more realistic, it shows the need for pollution coordination. Dynamic game theory seems to be a suitable way for dealing with such coordination needs.

Also in *macro-economic policy coordination*, dynamic game theory is a natural modeling framework. For instance, Tabellini [115] introduces a differential game model to study the strategic interaction between fiscal and monetary authorities, who both influence the dynamics of the government debt by controlling fiscal deficits and money creation, respectively. This model has been further deepened by van Aarle et al. [2] by analyzing Stackelberg open-loop equilibria of the game. Another application area concerns the introduction of the euro. This implies that national currencies are replaced by a common currency in the EMU. However, fiscal authorities remain controlled by the separate countries. Such a situation is put in a differential game context by van Aarle [1]. He analyses several equilibria in a differential game between national fiscal authorities and the ECB. Further results in this context are obtained in [47] and [3].

1.1.3 Uncertainty

A model is by definition an approximation of reality. In control engineering one uses a model of a plant to design a controller. This controller is based on the analysis of the model. It depends on the accuracy of the model whether the controller gives good performance or not. High accuracy of a model usually implies a high level of complexity. In economics, models can be used to coordinate strategic behavior between economic agents. Assumptions need to be made in order to be able to analyze the model. The more accurate the model is, the more effective the coordination is. Simplifying assumptions and parameter uncertainty in an engineering or economic model is called *model uncertainty*.

Model uncertainty received a great deal of attention in control engineering. The basic idea usually is to construct a set of models, each of which describes the same plant. Next, the controller is designed such as to yield good performance for each of these models. In this way the controller is robust with respect to model uncertainty within this set. Clearly, this concept can be worked out in many different ways. For instance one may assume that the set of models is generated by a stochastic process and choose a controller that minimizes the expected value of the performance criterion. Alternatively, one can choose a controller that optimizes the worst possible performance within the set of models. Such a controller is referred to as a *worst-case design*. Zero-sum game theory is a natural framework to construct worst-case designs. Indeed, in this context the set of models is generated by an unknown disturbance (taking values in a known set) entering the system. The uncertainty is modeled as the maximizing player who aims at maximizing the performance criterion by choosing an appropriate disturbance. The controller designer is the minimizing player who aims at minimizing the same performance criterion; in a minmax design he chooses the controller that optimizes the performance corresponding to the maximally disturbed model. This game-theoretic approach to robust controller design is in fact a way to treat the H_∞ control problem [9], which in itself deals with model uncertainty in the following sense. The system is assumed to be influenced by a disturbance and one looks for a controller minimizing the H_∞ norm of the transfer function from disturbance to output. H_∞ control theory was initiated in 1981 by Zames [126] and has extensively been studied in the 80s and the 90s; see for instance [9, 42, 50, 53, 57, 76, 118, 128].

1.1.4 Rationality and Planning-Horizon Lengths

In engineering a model is based on several physical laws, like gravity or conservation laws. In economics a model is often based on *rationality* of the economic agents involved. Whether people really act rationally is questionable. The climb to the summit of the Mount Everest on May 10, 1996, included the important rational rule: "if we have not reached the summit at 2 p.m., we return to camp 4". Some climbers did not act according to this rule and were still climbing after 2 p.m. Later in the afternoon, the good weather conditions suddenly changed into an horrible storm causing the death of some of the climbers. This example shows the need of verifying whether rationality can be used as an axiom in a model. A great deal of literature in the area of experimental economics is devoted to this issue; see for instance [38]. In engineering one can always put one's trust in physical laws. Whether or not one can rely on rational behavior in economics is uncertain.

In an optimal control problem or in a dynamic game the planning horizon is either finite or infinite. In engineering, an infinite-horizon problem can result from a minimization problem where the underlying function space consists of functions defined on the interval from zero to infinity; this is for instance the case in the infinite-horizon H_∞ optimal control problem. In such cases, the choice of an infinite horizon is motivated by mathematical convenience rather than by the specific application (robust controller design in the H_∞ case). In economics, it is usually the application that determines the planning horizon. An infinite horizon does not seem realistic; who takes the remaining of the existence of the universe into account in making strategical decisions? Usually authors discount future income or costs by implementing a discount factor in the model. Alternatively, one can model the problem with a finite horizon where its length is chosen in accordance with the application (e.g. policy makers are usually elected for a fixed period), or simply in accordance with the maximal period which seems relevant for the decision to be made. Once strategical decisions have been made, it is likely that the players (or decision maker in an optimal control problem) involved have an incentive to reoptimize their decisions after some time because new information has become available to them. Partly, this incentive is dealt with by the feedback Nash equilibrium concept due to its subgame perfectness [89, Chapter 9] (or strong time-consistency [8]); this property states that the optimal strategies on the time horizon $[0, T]$ are also optimal on any time horizon $[t, T]$. In the world around us, people are continuously updating their decisions by considering a new part of the future. One encounters such a behavior in important economic situations but also in real games like a game of chess. Indeed, a chess player continuously determines his action by thinking a number of moves ahead. After each move of his opponent, he extends his planning horizon, thus determining a new initial optimal action based on the new information which became available due to the move of his opponent. Clearly, we naturally deal with a bounded mind, i.e. in real life our decisions are inspired by a boundedly rational point of view. A moving horizon approach to model economic problems is a way to model this *bounded rationality*.

1.2 Objectives and a Review of the Thesis

Differential game theory would not exist if there was no optimal control theory. Many developments in differential game theory have been inspired by new achievements in system and control theory. The objectives of this thesis are to set up differential game models using successful developments in the control areas: moving horizon control theory, disturbance decoupling theory, and H_∞ control theory. Furthermore, the feedback Nash equilibrium for infinite-horizon games will be revised. Each of these topics will be studied in a linear quadratic (LQ) framework. A clear advantage is that each of the mentioned control areas has been extensively studied in an LQ setting. A disadvantage is its restrictiveness; in economic modeling one easily ends up in a non-linear framework. Nevertheless, as indicated earlier in this introduction, at the beginning of the 21st century LQ game theory has still many open problems and is extensively applied in economic applications, which justifies the choice of an LQ setting in this thesis.

In Chapter 2 we introduce some notations and terminologies that are used throughout the thesis. A few words are spent on the algebraic Riccati equation; in particular on its stabilizing solution. This solution plays quite an important role in Chapters 5, 6, and 7. In Section 2.3 the concept of a differential game will be made precise. Several technical elements are discussed in order to prevent misunderstandings later on in the thesis. Specifically, the following notions are discussed: state trajectories, control functions, information structures, strategy spaces, cost functions, coupled constraints, horizon lengths, stability, deterministic and stochastic disturbances, linearity, linear-quadratic aspects, time-invariance, and input-output structures.

In Chapter 3, a moving horizon solution concept is defined for the class of infinite-horizon differential games. In this concept the players have a feedback information pattern. At each time t, they determine an open-loop Nash equilibrium for a finite-horizon game. Due to the feedback information structure, the players have access to the state of the game at time t. This state is equal to the initial state of the finite-horizon game, which is required information to determine an open-loop Nash equilibrium. At time t the players only implement the initial control action corresponding to this equilibrium. By repeating this continuously in time, a feedback strategy defined on the infinite horizon results. The feedback information pattern makes the concept practically significant and the open-loop elements generate good analytic tractability.

Although the moving horizon concept will be defined for a general class of differential games (time-invariance is the only assumption we make), it will be analyzed in more detail in an LQ setting. One might expect here the use of fake Riccati equations; see for instance [40, 102]. This technique has been quite successful in LQ control theory to study stability of moving-horizon controllers[4]. However, since the technique owes its success to monotonicity properties of Riccati differential equations, a property which solutions of coupled systems of Riccati equations do not have, a generalization to a nonzero-sum game theoretic framework fails. Nevertheless, stability of the moving-horizon solution concept will be investigated. Under some symmetry conditions, a quadratic differential equation is derived for the closed-loop moving horizon matrix as a function of the length of the moving planning horizon. The analytic computability of the concept is addressed under the same symmetry conditions. The results are illustrated by an analysis of the scalar case and an economic example. Chapter 3 is self-contained and the following chapters can be read independently of this chapter. The subject of Chapter 3 has also been addressed by Kun [73]. Specifically, he replaced the local open-loop optimality by feedback optimality and obtained some stability results in a nonlinear context.

In Chapter 4, the starting point is an input-output (this terminology is made precise in the next chapter) differential game with an additive disturbance term in the differential equation. The basic question for each player here is: is it possible to construct a feedback strategy in such a way that my output is not influenced by the disturbance entering the system? Stated differently: can I find a feedback strategy which decouples my output from the disturbance? Clearly, this decoupling player has to take into account that the other players are also influencing his output. It will be assumed that these players are not cooperating to establish his decoupling purposes. The problem he faces is therefore called: the noncooperative disturbance decoupling problem. Necessary and sufficient conditions on the system parameters are derived under which the answer to the above question is positive. This is done for two different information structures: (i) the decoupling player is not able to observe other players' feedback strategies and (ii) he observes the other players' feedback strategies and uses this information to construct his own feedback strategy. Both solvability problems can be solved using standard techniques from geometric control theory. Particularly, the solvability problem under the second information structure is solved using the notion of robust controlled invariance [14]. The ensuing chapters can be read independently of Chapter 4.

Feedback Nash equilibria in infinite-horizon LQ differential games are usually determined from a set of coupled symmetric algebraic Riccati equations. In general, this set has multiple solutions. The equilibria obtained from these solutions are linear feedback strategies. A natural question is: are all equilibria obtained from these solutions? The answer to that question is "no": Tsutsui and Mino [120] have shown that there also exist nonlinear feedback strategies which constitute a Nash equilibrium. A second natural question therefore is: are all equilibria obtained from the solutions of the algebraic Riccati equations if the strategy spaces of the players are restricted to

[4] Nowadays, fake Riccati techniques are also used in a nonlinear context. Furthermore, not only stability, but also H_∞ robustness for nonlinear systems can be achieved by moving-horizon controllers; see for instance [85].

linear feedback strategies? The answer to that question is "yes", under the additional assumption that the set of admissible linear feedback strategies is restricted to strategies that stabilize the closed-loop system. This is the main result of Chapter 5. It is obtained without assuming the state weighting matrices to be positive semi-definite. The proof is based on a result for the corresponding one-player control problem: a linear, initial-state-independent, time-invariant, internally stabilizing state feedback law which minimizes a quadratic cost functional subject to a linear system constraint exists if and only if it is generated by the stabilizing solution of the corresponding algebraic Riccati equation. This will be shown first in Chapter 5 and can be considered as a contribution to LQ control theory. It is well-known that LQG and LQ control theory are linked through the certainty equivalence principle. Similarly, the feedback Nash equilibrium can be interpreted stochastically leading to the concept of a variance-independent feedback Nash equilibrium, which is formally defined in Section 5.5. It is shown that such equilibria are characterized by the solutions of the same set of algebraic Riccati equations.

Equally important questions as the ones posed in the previous paragraph are: how many solutions does the system of algebraic Riccati equations have? How can they be computed? In general, such problems are less extensively studied in this thesis than the relations between systems of Riccati equations and corresponding (robust) equilibria, which are extensively investigated in Chapters 5-7. Nevertheless, a discussion on the number and computability of solutions of the Riccati equations in the two-player scalar case is included in all of these chapters. In Chapter 5 it is shown that the solutions of the algebraic Riccati equations correspond to intersection points of hyperbolas in a certain half-plane. Furthermore, the number of intersection points may vary from zero to three and it is geometrically indicated how these situations, depending on the system parameters, occur.

In Chapters 6 and 7, model uncertainty in a differential game is expressed through a disturbance term in the differential equation. Following the lines of Chapter 5 the strategy spaces of the players are restricted to internally stabilizing linear feedback strategies. The theory to be developed in Chapters 6 and 7 can therefore conveniently be compared with the undisturbed case considered in Chapter 5. The robust feedback equilibrium concepts to be defined in these chapters are both inspired by H_∞ control theory. Loosely speaking, Chapters 6 and 7 generalize the disturbance attenuation problem with nonzero initial state and the soft-constrained differential game [9], respectively, to a nonzero-sum differential game setting.

Specifically, in Chapter 6 it is assumed that the disturbance is quadratically integrable on the positive real axis and that its norm is bounded by a given positive number r. A similar condition can be found in the H_∞ control problem (or disturbance attenuation problem) in the sense that the computation of the time-domain interpretation of the H_∞ norm requires the maximization of the quotient of the norms of the output and disturbance, where the maximization is carried out over the unit ball in L_2. However, in this criterion the initial state is assumed to be zero. The aim of Chapter 6 is to generalize the disturbance attenuation problem for nonzero initial state. For this generalization, an extensive study of the one-player optimal control problem is required. This problem consists of minimizing the maximally disturbed norm of the output of a linear system with nonzero initial state where the maximum is carried out over the ball with radius r in the disturbance space[5]. Chapter 6 starts with an intensive investigation of the zero-player problem, i.e. purely the maximization problem (with a fixed linear feedback controller), the analysis of which provides useful geometric insights for the rest of the chapter.

Chapter 7 differs from Chapter 6 in three respects: (i) the norm of the disturbance term is not restricted, (ii) the players weight the disturbance explicitly in their cost criteria with a negative term, and (iii) the state weighting matrix is not assumed to be positive semi-definite. It is the aim of the players to minimize the maximally disturbed cost criterion. The first two items cause the one-player problem to coincide with the soft-constrained differential game, which has extensively been studied in the literature. However, this problem seems to have been studied always under the assumption that the state weighting matrix is positive semi-definite. For that reason, a comprehensive analysis of the soft-constrained differential game is contained in Chapter 7. This material is used to define and analyze soft-constrained feedback Nash equilibria. A similar equilibrium concept in an open-loop finite-horizon context has been developed in [63].

[5] When the research of this problem was carried out, I was not aware of any reference in this direction. The closest reference seemed to be [32]. Later, [20] was brought to my attention, which is a closer reference. Chapter 6 has been written independently from this reference.

Chapter 2

Preliminaries

2.1 Notations, Terminologies, and Theorems

The following notations and terminologies are standard throughout the thesis.

(i) A symmetric $n \times n$ matrix P is called *positive definite* if $x^T P x > 0$ for all $x \in I\!R^n \setminus \{0\}$. This is denoted by $P > 0$. Similarly, P is called *positive semi-definite* if $x^T P x \geq 0$ for all $x \in I\!R^n$, and is denoted by $P \geq 0$.

(ii) The Euclidean length $\sqrt{a^T a}$ of a vector $a \in I\!R^n$ is denoted by $|a|$.

(iii) Let P be a positive semi-definite $n \times n$ matrix and let $a \in I\!R^n$, then

$$|a|_P := \sqrt{a^T P a}. \tag{2.1}$$

(iv) For a matrix $A \in I\!R^{m \times n}$ with columns $a_i \in I\!R^m$ for $i = 1, \ldots, n$, the vector $\operatorname{vec} A \in I\!R^{mn}$ is defined by

$$\operatorname{vec} A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}. \tag{2.2}$$

(v) Let $x_1, \ldots, x_N \in \mathcal{V}$ for some vector space $\mathcal{V}$, and let $j \in \{1, \ldots, N\}$. Then

$$\sum_{i \neq j}^{N} x_i := \left(\sum_{i=1}^{N} x_i \right) - x_j. \tag{2.3}$$

(vi) For an N-tuple $\gamma = (\gamma_1, \ldots, \gamma_N) \in \Gamma_1 \times \cdots \times \Gamma_N$ for given sets Γ_i, we write

$$\gamma_{-i}(\alpha) = (\gamma_1, \ldots, \gamma_{i-1}, \alpha, \gamma_{i+1}, \ldots, \gamma_N) \tag{2.4}$$

with $\alpha \in \Gamma_i$.

(vii) The space of $I\!R^k$-valued functions that are quadratically integrable on $(0, \infty)$ is denoted by $L_2^k(0, \infty)$. This space is a real Hilbert space with inner product

$$\langle v, w \rangle = \int_0^\infty v(t)^T w(t) dt, \quad v, w \in L_2^k(0, \infty). \tag{2.5}$$

The corresponding norm of $v \in L_2^k(0, \infty)$ is denoted by $||v||$, i.e. $||v|| = \sqrt{\langle v, v \rangle}$.

(viii) The space of $I\!R^k$-valued functions that are quadratically integrable on $(0, T)$ for all $T > 0$ is denoted by $L_{2,loc}^k(0, \infty)$.

(ix) Let $\mathcal{X}$ be a vector space, $\mathcal{Y}$ a normed space, $\mathcal{D} \subset \mathcal{X}$, and let T be an operator from $\mathcal{D}$ to $\mathcal{Y}$. If, for $x \in \mathcal{D}$ and $h \in \mathcal{X}$, the limit

$$\lim_{\alpha \to 0} \frac{T(x + \alpha h) - T(x)}{\alpha} \tag{2.6}$$

exists, it is called the *Gateaux differential of T at x with increment h*, and is denoted by $\delta T(x; h)$. If the limit (2.6) exists for each $h \in \mathcal{X}$, the operator T is called *Gateaux differentiable at x*. Assume further that $\mathcal{X}$ is a normed space and that $\mathcal{D}$ is an open subset of $\mathcal{X}$. If for fixed $x \in \mathcal{D}$ and each $h \in \mathcal{X}$ there exists a $y \in \mathcal{Y}$ such that (i) $h \mapsto y$ is linear and continuous, and (ii)

$$\lim_{||h|| \to 0} \frac{||T(x + h) - T(x) - y||}{||h||} = 0, \tag{2.7}$$

then T is called *Fréchet differentiable at x*, and y is called the *Fréchet differential of T at x with increment h* and is denoted by $y = \delta T(x; h)$ (following the literature [81, Section 7.2], we use the same symbols to denote the Gateaux and Fréchet differentials). If the Fréchet differential of T exists at x, then (i) it is unique, (ii) the Gateaux differential exists at x, and (iii) the Gateaux and Fréchet differentials are equal. Assume further that the operator T is Fréchet differentiable for each $x \in \mathcal{D}$. At a fixed point $x \in \mathcal{D}$, the Fréchet differential $\delta T(x; h)$ is, by definition, of the form $\delta T(x; h) = A_x h$, with A_x a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$. The correspondence $x \mapsto A_x$ defines a transformation from $\mathcal{D}$ into the normed linear space consisting of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. This transformation is called *Fréchet derivative* of T, and is denoted by ∂T. By definition, we have $\delta T(x; h) = \partial T(x) h$. Partial derivatives and differentials are denoted by ∂_i and δ_i, where the index refers to the corresponding argument.

(x) Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces. The kernel and image of a linear operator T from $\mathcal{V}$ to $\mathcal{W}$ are denoted by $\ker T$ and $\operatorname{im} T$, respectively.

(xi) The real and imaginary part of a number $z \in \mathbb{C}$ are denoted by $\operatorname{Re} z$ and $\operatorname{Im} z$, respectively.

(xii) A *two-player zero-sum game* is characterized by a triple $(\mathcal{X}, \mathcal{Y}, \varphi)$ where $\mathcal{X}$ and $\mathcal{Y}$ are sets and where φ is a function from the Cartesian product $\mathcal{X} \times \mathcal{Y}$ to $\mathbb{R}$. The set $\mathcal{X}$ ($\mathcal{Y}$) is the strategy set of the minimizing (maximizing) player, respectively. For a given pair of strategies $(x, y) \in \mathcal{X} \times \mathcal{Y}$ the outcome of the game is $\varphi(x, y)$. The minimizing (maximizing) player tries to minimize (maximize) this outcome by choosing an appropriate $x \in \mathcal{X}$ ($y \in \mathcal{Y}$). The *lower value* and *upper value* of the game are defined by

$$\underline{\varphi} := \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \varphi(x, y);$$
$$\overline{\varphi} := \inf_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} \varphi(x, y),$$

respectively. It is easily seen that $\underline{\varphi} \leq \overline{\varphi}$, which justifies the terminology. If the lower value equals the upper value, the common value $\varphi^* := \underline{\varphi} = \overline{\varphi}$ is called the *value* of the game. A pair of strategies $(\overline{x}, \overline{y}) \in \mathcal{X} \times \mathcal{Y}$ is called a *saddle-point solution* for the game if

$$\varphi(\overline{x}, y) \leq \varphi(\overline{x}, \overline{y}) \leq \varphi(x, \overline{y}) \text{ for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}.$$

It is easily seen that if a saddle-point solution $(\overline{x}, \overline{y})$ exists, upper and lower value are equal and the value of the game equals $\varphi(\overline{x}, \overline{y})$. For a general account of static zero-sum game theory we refer to [98] and for an account of zero-sum dynamic game theory we refer to [9, 10].

(xiii) A matrix $A \in \mathbb{R}^{n \times n}$ is called *stable* if all its eigenvalues are in the open left-half plane.

(xiv) For matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, the set $\mathcal{F}$ is defined by

$$\mathcal{F} := \left\{ F \in \mathbb{R}^{m \times n} \,\middle|\, A + BF \text{ stable} \right\}. \tag{2.8}$$

(xv) For a number $N \in \mathbb{N}$ and matrices $A \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m_i}$ ($i = 1, \ldots, N$), the set $\mathcal{F}_N$ is defined by

$$\mathcal{F}_N := \left\{ (F_1, \ldots, F_N) \,\middle|\, A + \sum_{j=1}^{N} B_j F_j \text{ stable} \right\}. \tag{2.9}$$

(xvi) Let $\mathcal{X}$ be a Hilbert space. The closed ball with center $0 \in \mathcal{X}$ and radius $r > 0$ is denoted by B_r, i.e.

$$B_r := \{x \in \mathcal{X} | \|x\| \leq r\}. \tag{2.10}$$

(xvii) The expectation operator is denoted by E.

(xviii) A number $z_0 \in \mathbb{C}$ is called an *invariant zero* of the matrix quadruple (A, B, C, D) if

$$\begin{bmatrix} A - z_0 I & B \\ C & D \end{bmatrix} < \operatorname{normrank} \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix}$$

where the *normal rank* of a polynomial matrix $Q(z)$ is defined by

$$\operatorname{normrank} Q(z) = \max\{\operatorname{rank} Q(z) | z \in \mathbb{C}\}.$$

The following two standard results are used throughout the thesis.

Theorem 2.1.1 *Consider a real Hilbert space $\mathcal{X}$, a bounded self-adjoint linear operator T on $\mathcal{X}$, and an element $x_0 \in \mathcal{X}$. Let the quadratic functional $f : \mathcal{X} \to \mathbb{R}$ be defined by*

$$f(x) = \langle x, Tx \rangle + 2 \langle x, x_0 \rangle .$$

Then, f has a maximum if and only if $-T$ is positive semi-definite and $x_0 \in \operatorname{im} T$. If these conditions hold, f attains its maximal value at all points $\overline{x}$ with $T\overline{x} = -x_0$. Furthermore $f(\overline{x}) = \langle \overline{x}, x_0 \rangle$.

Lemma 2.1.2 (see e.g. [88, Exercise 6.10]) *Let $x \in L_2^k(0, \infty)$ be differentiable and denote the derivative by $\dot{x}$. If $\dot{x} \in L_2^k(0, \infty)$, then $x(t) \to 0$ as $t \to \infty$.*

2.2 The Algebraic Riccati Equation

Quite an important role in this thesis is played by the algebraic Riccati equation. This equation has extensively been studied in the literature; see for instance [21, 23, 76, 118, 123, 128]. Here we briefly review some aspects that are important for this thesis.

Let A, Q, and R be real $n \times n$ matrices with Q and R symmetric. The *algebraic Riccati equation* (ARE) is the matrix equation

$$Q + A^T X + XA + XRX = 0. \tag{2.11}$$

Associated with this Riccati equation is the $2n \times 2n$ Hamiltonian matrix

$$H := \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix}. \tag{2.12}$$

Of particular interest are solutions X of (2.11) with the property that $A + RX$ is stable. Next, we define an operator from a certain set of $2n \times 2n$ Hamiltonian matrices to the set of $n \times n$ matrices, with the property that each image is a solution of (2.11) that has this stability property.

Assume that the matrix H has no eigenvalues on the imaginary axis. Denote the spectral subspaces corresponding to the eigenvalues in the open left-half plane and the open right-half plane by $\mathcal{X}_-(H)$ and $\mathcal{X}_+(H)$, respectively. Let X_1 and X_2 be real $n \times n$ matrices such that

$$\mathcal{X}_-(H) = \mathrm{im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{2.13}$$

If X_1 is regular, or, equivalently, if the two subspaces

$$\mathcal{X}_-(H) \quad \text{and} \quad \mathrm{im} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{2.14}$$

are complementary, we can define $X := X_2 X_1^{-1}$. The correspondence $H \mapsto X$ is called the *Riccati operator* and is denoted by Ric. The domain $\mathrm{dom}(\mathrm{Ric})$ of this operator consists of Hamiltonian matrices H with two properties: H has no eigenvalues on the imaginary axis and the two subspaces in (2.14) are complementary. The image $X = \mathrm{Ric}(H)$ has the following properties.

Theorem 2.2.1 (see e.g. [128, Theorem 13.5]) *Suppose $H \in \mathrm{dom}(\mathrm{Ric})$ and $X = \mathrm{Ric}(H)$. Then*

(i) X is real symmetric;

(ii) X satisfies the algebraic Riccati equation (2.11)*;*

(iii) the matrix $A + RX$ is stable.

One can easily show that a matrix X with these properties is unique. If this matrix X exists, it is called *the stabilizing solution* of the ARE. The existence of the stabilizing solution of the ARE can be verified by checking whether the corresponding Hamiltonian matrix has no purely imaginary eigenvalues, and whether the complementarity condition (2.14) holds. This second condition can be replaced by a rank condition on the matrix sign of a certain matrix ([76, Theorem 22.4.1] or [77, p. 175]). An extensive literature on algorithms for accurately computing the matrix sign exists, and one finds a comprehensive list of references in the review paper of Laub [77].

2.3 Differential Games

We take the definition of a differential game given by Başar and Olsder [10, Definition 5.5] is a starting point for the introduction of a general framework for the differential games considered in this thesis. A modified version of this definition is given below. The modification is motivated by the three types of information structures that are considered in this thesis.

Definition 2.3.1 An N-player finite-horizon differential game involves the following:

(i) An index set $\mathbf{N} = \{1, \ldots, N\}$ called the *set of players*.

(ii) A time interval $[t_0, t_1]$ which is specified a priori and which denotes the duration of the game.

(iii) A set $\mathcal{S}_0$ consisting of functions from $[t_0, t_1]$ to S_0 with S_0 a subset of $I\!R^n$. This set is called the *trajectory space* of the game. Its elements are the permissible *state trajectories* of the game.

(iv) For each $i \in \mathbf{N}$, a set $\mathcal{U}_i$ consisting of functions from $[t_0, t_1]$ to S_i with S_i a subset of $I\!R^{m_i}$. This set is called the *control space* of player i. Its elements are the permissible *control functions* for player i. The value $u_i(t)$ is called the *action* of player i at time t.

(v) A differential equation

$$\dot{x} = f(t, x, u_1, \ldots, u_N), \quad x(0) = x_0. \tag{2.15}$$

The solution $x \in \mathcal{S}_0$ is the state trajectory of the game corresponding to the N-tuple of control functions $(u_1, \ldots, u_N) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_N$ and the initial state x_0.

(vi) A function η_i defined on $[t_0, t_1]$ for each $i \in \mathbf{N}$ which determines the state information available to player i at each time $t \in [t_0, t_1]$. Specification of η_i characterizes the *information structure* of player i. In this thesis we use three different information structures:

(a) if $\eta_i : [t_0, t_1] \to I\!R^n$, $\eta_i(t) = x_0$, the information structure of player i is said to be an *open-loop* pattern;

(b) if $\eta_i : [t_0, t_1] \to I\!R^n$, $\eta_i(t) = x(t)$, the information structure of player i is said to be a *feedback* pattern;

(c) if $\eta_i : [t_0, t_1] \to I\!R^n \times I\!R^n$, $\eta_i(t) = (x_0, x(t))$, the information structure is said to be a *memoryless perfect state* pattern.

For more information structures we refer to [10, Definition 5.6].

(vii) For each $i \in \mathbf{N}$, a set Γ_i of mappings $\gamma_i : (t, \eta_i(t)) \mapsto u_i(t)$ called the *strategy space* of player i. Its elements are the permissible *strategies* for player i.

(viii) For each $i \in \mathbf{N}$, two functionals $q_i : S_0 \to I\!R$ and $g_i : [t_0, t_1] \times S_0 \times S_1 \times \cdots \times S_N \to I\!R$, determining a functional $J_i : \Gamma_1 \times \cdots \times \Gamma_N \to I\!R$ as follows

$$J_i(\gamma_1, \ldots, \gamma_N) = \int_{t_0}^{t_1} g_i(t, x(t), u_1(t), \ldots, u_N(t))dt + q_i(x(t_1)). \tag{2.16}$$

The functional J_i is called the *cost function* of player i. □

An N-player finite-horizon differential game is well defined if (i) the function f and the strategy spaces Γ_i are such that the differential equation (2.15) admits a unique solution for each N-tuple $(u_1, \ldots, u_N)$ with $u_i(t) = \gamma_i(t, \eta_i(t))$, $\gamma_i \in \Gamma_i$, and (ii) the functionals q_i and g_i are such that the integral (2.16) exists for each permissible N-tuple of strategies and for each $i \in \mathbf{N}$. Sufficient conditions for the first condition to hold can be found in [10, Theorem 5.1]. All the differential games considered in this thesis are well defined.

2.3.1 Generalizations and Special Structures

(i) Let

$$\Gamma := \Gamma_1 \times \cdots \times \Gamma_N. \tag{2.17}$$

The set of permissible strategies can be further restricted to a subset $\Gamma_0 \subset \Gamma$ which cannot be written as the Cartesian product of individual strategy spaces. Such a game is called a *coupled-constraint differential game*. In this situation the players cannot choose strategies independently. One can think of coupled constraints as a cooperative aspect in a noncooperative game. Although each player has his own objective, there should be a coordination between the players to choose the N-tuples of strategies in the set Γ_0. Coupled constraints were studied in the context of static games in [106]. Recently, the concept of coupled constraints has been applied to an environmental compliance problem where agents are enticed to obey pollution constraints jointly rather than separately [70].

(ii) An *infinite-horizon* differential game is defined just like a finite-horizon game in Definition 2.3.1 with $t_1 = \infty$ and $q_i = 0$ for all $i \in \mathbf{N}$. An N-player infinite-horizon differential game is called *stable* if

(a) $\Gamma \neq \emptyset$;

(b) for each permissible N-tuple of strategies in Γ the state trajectory converges to a steady state value for all initial states $x_0 \in S_0$.

There exist situations where the state converges to a steady state value in a Nash equilibrium, and where a player can improve unilaterally to a strategy resulting in a state trajectory that diverges to infinity. By imposing a coupled-constraint condition on the strategy spaces the game can be made stable and thus such a deviation is not allowed. This condition can be interpreted as a coordination between the players to achieve the common stability objective.

(iii) A differential game is called *disturbed* if the right-hand side of the differential equation (2.15) has an extra term w, which is uncontrollable for all the players. This term w is called the *disturbance*. Possibly, also the functions g_i (see (2.16)) depend on w. We call a differential game *deterministically disturbed* if the disturbance w is an element of some topological function space $\mathcal{W}$[1]. This space should be such that the differential game is well-defined for each $w \in \mathcal{W}$; see the discussion following Definition 2.3.1. We call a differential game *stochastically disturbed* if the disturbance w is a stochastic process.

(iv) A (disturbed) differential game is called *linear* if $S_i = \mathbb{R}^{m_i}$ and

$$f(t, x, u_1, \ldots, u_N, w) = A(t)x + \sum_{j=1}^{N} B_j(t)u_j + E(t)w \tag{2.18}$$

where $A(t) \in \mathbb{R}^{n\times n}$, $B_j(t) \in \mathbb{R}^{n\times m_j}$, and $E(t) \in \mathbb{R}^{n\times q}$ are matrices defined on $[t_0, t_1]$ with continuous entries. Note that the game is disturbed if $E(t) \neq 0$. A strategy corresponding to a feedback information structure is called linear if

$$\gamma_i(t, x) = F_i(t)x \tag{2.19}$$

where $F_i(t) \in \mathbb{R}^{m_i\times n}$ is a matrix defined on $[t_0, t_1]$ with continuous entries. We call a strategy corresponding to a memoryless perfect state information structure linear if

$$\gamma_i(t, x_0, x) = F_i(t, x_0)x \tag{2.20}$$

where $F_i(t, x_0) \in \mathbb{R}^{m_i\times n}$ is a matrix defined on $[t_0, t_1] \times \mathbb{R}^n$ with continuous entries.

(v) A (disturbed) differential game is called *linear-quadratic* if the game is linear and if

$$g_i(t, x, u_1, \ldots, u_N, w) = x^T Q_i x + \sum_{j=1}^{N} u_j^T R_{ij}(t)u_j + w^T V_i(t)w; \tag{2.21}$$

$$q_i(x) = x^T Q_{if} x \tag{2.22}$$

[1] In this thesis, we always take $\mathcal{W} = L_2^q(0, \infty)$.

where $Q_i(t), Q_{if} \in I\!R^{n\times n}$, $R_{ij}(t) \in I\!R^{m_j\times m_j}$, $V_i(t) \in I\!R^{q\times q}$ are symmetric matrices defined on $[t_0, t_1]$ with continuous entries. Furthermore $R_{ii} > 0$ for all $i \in \mathbf{N}$. The matrices R_{ij} for $i \neq j$ are called *cross-control* matrices. The matrices Q_i and R_{ii} are called *state-weighting* and *control-weighting* matrices, respectively.

(vi) A differential game is called *time-invariant* if the functions f and g_i do not depend on the time variable t. Similarly, the strategy of player i is called time-invariant if it does not depend on the time variable t.

(vii) We say that a differential game is of the *input-output* type if an output function is associated to each player. The output function of player i is $I\!R^{p_i}$-valued and is denoted by z_i and depends in general on the time variable t, the state variable x and the control variables $u_1, \dots, u_N$. If the game is linear time-invariant, z_i has the form

$$z_i = C_i x + \sum_{i=1}^{N} D_{ij} u_j \tag{2.23}$$

where $C_i \in I\!R^{p_i\times n}$ and $D_{ij} \in I\!R^{p_i\times m_j}$ are constant matrices. If the outputs z_i are elements of a normed space, one can represent cost functions as $||z_i||^2$. If the game is disturbed, z_i can also depend on the disturbance w; however such a situation does not occur in this thesis.

Chapter 3

Moving Horizon Strategies

3.1 Introduction

Because of its practical relevance, the feedback Nash equilibrium is a popular solution concept in the area of dynamic games [3, 35, 86, 100]. Unfortunately, this concept is usually not analytically tractable and authors often use numerical techniques to study feedback Nash equilibria in a dynamic game. The open-loop Nash equilibrium on the other hand enables one to study a dynamic game more analytically; especially in the class of linear-quadratic games [1, 2, 47, 44, 45]. However, this concept has less practical relevance due to its limited information structure. In many applications the two solution concepts are both computed, either analytically or numerically, and the corresponding strategies are compared to each other [93, 48, 115].

Another basic question in a dynamic game model is the length of the planning horizon on which the players base their decisions. Is it finite or infinite? And, if it is finite, what is its length? These questions are not always easy to answer. A suitable approach dealing with this problem in a one-player context seems to be the model of *rolling horizon decision making* [12, 109, 69, 71]. In this approach the decision maker bases his decision on a forecast over a finite horizon. In the next period he takes an additional period of the future into account in order to make the next optimal decision. This procedure is repeated every period which justifies the term rolling horizon. The term horizon refers to the number of periods in the future for which the forecast is made. A similar approach has been developed in the area of repeated games by Jéhiel [64, 65]. The rolling horizon concept does not seem to have received any attention in the area of differential games. It is the purpose of the present chapter to develop some ideas in this direction. Specifically, a concept, called the *moving horizon solution concept*, is developed for the class of infinite-horizon time-invariant differential games where the information structure of each player is a feedback pattern; thus making this concept practically relevant. The strategies of the players are at each point in time based on a finite-horizon differential game with an open-loop information pattern, which makes the concept analytically tractable.

The moving horizon solution concept for a differential game as introduced in this chapter is also inspired by the development of *model predictive control* [5] in the last few decades. This method has become quite popular in controller design problems for industrial processes. In model predictive control one optimizes forecasts of process behavior with a finite horizon process model. Only the first input of the optimal input sequence is injected to the plant and the problem is solved again at the next time interval using updated process measurements and a shifted horizon. Authors often use the synonymous terms *receding horizon control* or *moving horizon control* to refer to such a control scheme. The origin of this methodology goes back to a brief note of Kleinman [67] who constructed a class of feedback controllers to stabilize a linear system which resembles a moving horizon controller. The development of moving horizon control has been carried out and generalized in many directions. To name just a few: systems with constraints both on state and input, time-varying systems [74], and non-linear systems [90, 91, 55, 56]; also robustness properties have been studied [41, 11]. An extensive literature review can be found in [5]. Moving horizon control has also been applied in zero-sum game theory [75, 85], thus connecting it to H_∞ control theory. However, no literature seems available in the area of nonzero-sum game theory. This chapter is a first attempt in that direction.

A choice between a finite planning horizon (and its length) or an infinite horizon is avoided by the moving horizon approach. Nevertheless one still deals with the choice of the moving horizon length. In some cases the application could resolve this issue; for instance in production planning problems these lengths are usually known. In such a case, the moving horizon length is an exogenous parameter. Alternatively, if such a natural choice is not possible, it could be preferable to make the moving horizon length an endogenous parameter, i.e. to let it result from a certain optimization problem. In the present chapter a method is presented to choose the moving horizon length in such a way that the distance between the moving horizon solution and a feedback Nash equilibrium is minimized. Since in general feedback Nash equilibria are not unique [46, 122], this additionally provides a refinement of this equilibrium concept that can be used to select a unique feedback Nash equilibrium.

The outline of this chapter is as follows. The moving horizon solution concept is defined for N-player time-invariant infinite-horizon differential games in Section 3.2. The concept is studied in more detail in the class of linear-quadratic games in Section 3.3. In particular a necessary and sufficient condition is presented for a time-invariant moving horizon solution to exist uniquely. Special attention is paid to the scalar case in Section 3.3.2. Suggestions about how to make the moving horizon length an endogenous parameter are given in Section 3.3.3. The theory and analytic tractability of the moving horizon concept are illustrated by an economic example in Section 3.4. This example is a government debt stabilization differential game where the players are a fiscal and a monetary authority. The chapter ends with some concluding remarks in Section 3.5.

3.2 The Moving Horizon Solution Concept

Following the lines of moving horizon control, the players in a dynamic game determine optimal strategies based on a finite horizon at each point in time. However, they only implement the initial control action. By repeating this continuously in time, strategies defined on the infinite time horizon result. In this section a solution concept based on this notion is defined in the class of N-player infinite-horizon time-invariant differential games in which the information structure of each player is a feedback pattern. This concept will formally be defined in Definition 3.2.1 below, which requires some preliminary work.

Consider an N-player infinite-horizon time-invariant differential game in which the information structure of each player is a feedback pattern; see Section 2.3 for the notation and terminology. The specification of such a game involves a differential equation of the form

$$\dot{x} = f(x, \gamma_1(t, x), \dots, \gamma_N(t, x)). \tag{3.1}$$

The strategy space Γ_i of player i is defined as the subset of all maps from $[0, \infty) \times I\!R^n$ to $I\!R^{m_i}$ such that for each $(\gamma_1, \dots, \gamma_N) \in \Gamma_1 \times \dots \times \Gamma_N$ there exists a unique state trajectory on the infinite horizon for each initial state. Let the cost function of player i be given by

$$J_i(\gamma_1, \dots, \gamma_N, x_0) = \int_0^\infty g_i(x(t), \gamma_1(t, x(t)), \dots, \gamma_N(t, x(t)))dt,$$

where $\gamma_j \in \Gamma_j$ for $j = 1, \dots, N$, and x is the solution of (3.1) with $x(0) = x_0$. In the sequel, this infinite-horizon differential game is referred to as the global game. In addition to the global game, consider at each point in time t_0 a finite-horizon differential game on the time horizon $[t_0, t_0 + L]$ for some fixed horizon length L. It is assumed that the information structure of the players is an open-loop pattern for each such differential game. In the game on the time horizon $[t_0, t_0 + L]$, the strategies of the players are denoted by

$$\hat{\gamma}_i^{t_0} : [t_0, t_0 + L] \times I\!R^n \to I\!R^{m_i}, \quad \hat{\gamma}_i^{t_0} : (t, x(t_0)) \mapsto \hat{u}_i^{t_0}(t), \quad t_0 \le t \le t_0 + L.$$

Note that the initial state on which these open-loop strategies depend, is exactly the state $x(t_0)$ of the global game at time t_0. The strategy space $\hat{\Gamma}_i^{t_0}$ of player i in the game on the time horizon $[t_0, t_0 + L]$ is defined as the subset of maps from $[t_0, t_0 + L] \times I\!R^n$ to $I\!R^{m_i}$ such that for each $(\hat{\gamma}_1, \ldots, \hat{\gamma}_N) \in \hat{\Gamma}_1^{t_0} \times \cdots \times \hat{\Gamma}_N^{t_0}$, there exists a unique state trajectory on the interval $[t_0, t_0 + L]$ for each initial state $x(t_0)$. Denote this state trajectory by $\hat{x}^{t_0}$. Then, $\hat{x}^{t_0}$ satisfies the differential equation

$$\dot{\hat{x}}^{t_0}(\tau) = f\left(\hat{x}^{t_0}(\tau), \hat{u}_1^{t_0}(\tau), \ldots, \hat{u}_N^{t_0}(\tau)\right), \quad t_0 \leq \tau \leq t_0 + L, \tag{3.2}$$

and initial condition $\hat{x}^{t_0}(t_0) = x(t_0)$. Furthermore, the cost function of player i is given by

$$\hat{J}_i^{t_0}\left(\hat{\gamma}_1^{t_0}, \ldots, \hat{\gamma}_N^{t_0}, x(t_0)\right) = \int_{t_0}^{t_0+L} g_i\left(\hat{x}^{t_0}(\tau), \hat{u}_1^{t_0}(\tau), \ldots, \hat{u}_N^{t_0}(\tau)\right) dt,$$

where $\hat{\gamma}_j^{t_0} \in \hat{\Gamma}_j^{t_0}$ for $j = 1, \ldots, N$, and $\hat{x}^{t_0}$ is the solution of (3.2) with initial state $x(t_0)$. We are now able to formally define the moving horizon solution concept.

Definition 3.2.1 Let $L > 0$. Let for each $t_0 > 0$, the set of N strategies $\hat{\gamma}_i^{t_0} \in \hat{\Gamma}_i^{t_0}$ for $i = 1, \ldots, N$, constitute an open-loop Nash equilibrium for the game on the time horizon $[t_0, t_0 + L]$. Then the set of N strategies $\gamma_i \in \Gamma_i$, defined by

$$\gamma_i(t, x) := \hat{\gamma}_i^t(t, x), \quad i = 1, \ldots, N, \tag{3.3}$$

is called a *moving horizon solution* for the global game with moving horizon length L. Individual strategies are called *moving horizon strategies*. □

In a moving horizon solution the players face a finite-horizon game at each point in time, determine an open-loop Nash equilibrium for this game, and only implement the initial equilibrium control action in the global game. In general there may exist multiple open-loop Nash equilibria for the finite horizon games. If this is the case, Definition 3.2.1 allows the players to switch between different equilibria as time evolves. It may be expected that such a switching behavior leads to unstable dynamics. In what follows, a moving horizon solution is constructed that excludes the possibility of switching between different equilibria. In this construction, the time-invariance of the functions f and g_i plays an important role. The constructed moving horizon solution is also time-invariant.

Let $\hat{\gamma}_1^0, \ldots, \hat{\gamma}_N^0$ be a set of open-loop strategies for the finite horizon game on the time horizon $[0, L]$. Let $\xi \in I\!R^n$ be the initial state of this game. Denote the corresponding state trajectory and costs by $\hat{x}^0$ and $\hat{J}_i^0$, respectively. Let $t_0 > 0$ and define the open-loop strategies $\hat{\gamma}_i : [t_0, t_0 + L] \times I\!R^n \to I\!R^{m_i}$ for the game on the time horizon $[t_0, t_0 + L]$ by $\hat{\gamma}_i(\tau, x) := \hat{\gamma}_i^0(\tau - t_0, x)$. Denote the state trajectory corresponding to these strategies and initial state ξ by $\hat{x}$. By construction, we have

$$\dot{\hat{x}}^0(\tau) = f\left(\hat{x}^0(\tau), \hat{\gamma}_1^0(\tau, \xi), \dots, \hat{\gamma}_N^0(\tau, \xi)\right), \quad 0 \leq \tau \leq L, \quad \hat{x}^0(0) = \xi$$

and

$$\dot{\hat{x}}(\tau) = f\left(\hat{x}(\tau), \hat{\gamma}_1(\tau, \xi), \dots, \hat{\gamma}_N(\tau, \xi)\right), \quad t_0 \leq \tau \leq t_0 + L, \quad \hat{x}(t_0) = \xi.$$

Since f is time-invariant it follows that $\hat{x}(\tau) = \hat{x}^0(\tau - t_0)$ for all $\tau \in [t_0, t_0 + L]$. This implies that

$$\begin{aligned}
\hat{J}_i(\hat{\gamma}_1, \dots, \hat{\gamma}_N, \xi) &= \int_{t_0}^{t_0+L} g_i(\hat{x}(\tau), \hat{\gamma}_1(\tau, \xi), \dots, \hat{\gamma}_N(\tau, \xi)) d\tau = \\
&= \int_{t_0}^{t_0+L} g_i\left(\hat{x}^0(\tau - t_0), \hat{\gamma}_1^0(\tau - t_0, \xi), \dots, \hat{\gamma}_N^0(\tau - t_0, \xi)\right) d\tau = \\
&= \int_0^L g_i\left(\hat{x}^0(\tau), \hat{\gamma}_1^0(\tau, \xi), \dots, \hat{\gamma}_N^0(\tau, \xi)\right) d\tau = \hat{J}_i^0.
\end{aligned}$$

The latter equality follows from the time-invariance of the function g_i. Hence, the costs of player i in the game on the horizon $[0, L]$ corresponding to the strategies $\hat{\gamma}_1^0, \dots, \hat{\gamma}_N^0$ and initial state ξ are equal to his costs in the game on the horizon $[t_0, t_0 + L]$ corresponding to the translated strategies $\hat{\gamma}_1, \dots, \hat{\gamma}_N$ and initial state ξ, for each point in time t_0. This observation implies the following result.

Proposition 3.2.2 *Let $L > 0$. Let $\hat{\gamma}_1, \dots, \hat{\gamma}_N$ be an open-loop Nash equilibrium for the game on the time-horizon $[0, L]$. Then, the set of N time-invariant feedback strategies $\gamma_i : I\!R^n \to I\!R^{m_i}$ defined by $\gamma_i(x) := \hat{\gamma}_i(0, x)$ is a time-invariant moving horizon solution for the global game with moving horizon length L.*

Only time-invariant moving horizon solutions as generated by the previous theorem are considered in this thesis. The game on the time horizon $[0, L]$ in which the players determine an open-loop Nash equilibrium is referred to as the local game. A time-invariant moving horizon strategy of player i corresponding to a moving horizon solution generated by a fixed open-loop Nash equilibrium of the local game typically depends on L, and will therefore be denoted by $\gamma_i^{MH}(x; L)$.

Remark 3.2.3 In finite horizon problems cost functions can be provided with a terminal penalty, which is obviously not taken into account by the moving horizon solution concept. Although it would not make the analysis more difficult, it does not seem very realistic to do this in the moving horizon solution concept, since the players never reach the endpoint of the local game. □

Remark 3.2.4 An important issue that needs to be studied is stability, i.e. does a moving horizon solution stabilize the closed-loop system? The planning horizon length L of the local game plays an important role in this. It is to be expected that especially for small values of L stability problems may arise. □

Remark 3.2.5 Important notions which are often dealt with in the literature are weak and strong time consistency; see for instance [8] or [10, Section 5.6]. Time-invariant moving horizon solutions generated by Proposition 3.2.2, which are time-invariant feedback strategies, are clearly strongly time consistent. An important underlying assumption is that such moving horizon solutions are generated by a fixed open-loop Nash equilibrium for the local game. □

Remark 3.2.6 It is important to realize that the moving horizon concept is not defined as an equilibrium concept. It is a method to play the game which may resemble reality better than control strategies based directly on the existing equilibrium concepts. □

3.3 Moving Horizon Control in LQ Games

In this section the moving horizon solution concept is analyzed in more detail in the subclass of two-player linear-quadratic differential games. Such a game is specified by

$$f(x, u_1, u_2) = Ax + B_1u_1 + B_2u_2, \tag{3.4}$$

$$g_1(x, u_1, u_2) = x^TQ_1x + u_1^TR_{11}u_1 + u_2^TR_{12}u_2, \tag{3.5}$$

$$g_2(x, u_1, u_2) = x^TQ_2x + u_1^TR_{21}u_1 + u_2^TR_{22}u_2, \tag{3.6}$$

with A, B_i, Q_i, and R_{ij} constant matrices of appropriate dimensions. We assume that $Q_i \geq 0$, $R_{11} = R_{22} = I$, and $R_{12} = R_{21} = 0$.

3.3.1 General Formulation

As indicated in the paragraph following Proposition 3.2.2, we consider time-invariant moving horizon solutions generated by open-loop Nash equilibria for the local game on the horizon $[0, L]$.

Engwerda [45, Theorem 1] has shown that a unique open-loop Nash equilibrium exists for each initial state if and only if a certain matrix is invertible. In fact, if this condition holds, the unique open-loop Nash equilibrium is given by $\hat{\gamma}_i(\tau, \xi) = -B_i^T \psi_i(\tau)$, where the initial state is denoted by ξ. Here, the functions ψ_1 and ψ_2 follow from the two-point boundary value problem

$$\frac{d}{d\tau}\begin{bmatrix} \hat{x}(\tau) \\ \psi_1(\tau) \\ \psi_2(\tau) \end{bmatrix} = -M \begin{bmatrix} \hat{x}(\tau) \\ \psi_1(\tau) \\ \psi_2(\tau) \end{bmatrix}, \quad \hat{x}(0) = \xi, \quad \psi_1(L) = \psi_2(L) = 0, \tag{3.7}$$

with matrix M defined by

$$M := \begin{bmatrix} -A & S_1 & S_2 \\ Q_1 & A^T & 0 \\ Q_2 & 0 & A^T \end{bmatrix}, \quad S_i := B_i B_i^T. \tag{3.8}$$

If the local game has a unique open-loop Nash equilibrium, the corresponding time-invariant moving horizon solution is clearly the unique moving horizon solution for the global time-invariant game on the infinite horizon. This observation gives a necessary and sufficient condition for the existence of a unique moving horizon solution in the subclass of two-player linear-quadratic differential games. This condition is formulated in the following theorem.

Theorem 3.3.1 *Consider the two-player linear-quadratic infinite-horizon time-invariant differential game*

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad J_i = \int_0^\infty \left(x^T Q_i x + u_i^T u_i \right) dt$$

with $Q_i \geq 0$. Let the matrices S_i and M be defined by (3.8). *Furthermore, define for each $L > 0$ the $n \times n$ matrices $H(L)$, $G_1(L)$, and $G_2(L)$ by*

$$\begin{bmatrix} H(L) \\ G_1(L) \\ G_2(L) \end{bmatrix} := e^{LM} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}.$$

Then there exists a unique moving horizon solution with horizon length L if and only if the matrix $H(L)$ is regular. If this condition holds, the moving horizon strategies are given by

$$\gamma_i^{MH}(x; L) = -B_i^T G_i(L) H^{-1}(L) x, \quad i = 1, 2. \tag{3.9}$$

Proof The uniqueness of the moving horizon solution follows from the fact that the open-loop Nash equilibrium is unique. According to Proposition 3.2.2, we have $\gamma_i^{MH}(x;L) = \hat{\gamma}_i(0,x) = -B_i^T\psi_i(0)$. The boundary value problem (3.7) implies that

$$\begin{bmatrix} \hat{x}(\tau) \\ \psi_1(\tau) \\ \psi_2(\tau) \end{bmatrix} = e^{-ML} \begin{bmatrix} \xi \\ \psi_1(0) \\ \psi_2(0) \end{bmatrix}.$$

Taking $\tau = L$, multiplying the left-hand and the right-hand side by e^{ML}, and using the boundary conditions $\psi_1(L) = \psi_2(L) = 0$ yields

$$\begin{bmatrix} \xi \\ \psi_1(0) \\ \psi_2(0) \end{bmatrix} = \begin{bmatrix} H(L) \\ G_1(L) \\ G_2(L) \end{bmatrix} \hat{x}(L).$$

If $H(L)$ is regular, then $\hat{x}(L) = H(L)^{-1}\xi$. From this we see that $\psi_i(0) = G_i(L)H(L)^{-1}\xi$, which implies (3.9). □

It is well-known that open-loop Nash equilibria for finite-horizon LQ differential games are also generated by the solution of a set of asymmetric Riccati differential equations. If this solution exists on the complete interval $[0, L]$, the moving horizon solution is also generated by this solution. This property is formulated in the following corollary. In order to indicate the difference with differentiation with respect to t, differentiation with respect to L is denoted by a prime.

Corollary 3.3.2 *Let $L_0 > 0$. Assume that the solution (P_1, P_2) of the open-loop Riccati differential equations*

$$P_1' = Q_1 + A^T P_1 + P_1 A - P_1 S_1 P_1 - P_1 S_2 P_2, \tag{3.10}$$
$$P_2' = Q_2 + A^T P_2 + P_2 A - P_2 S_1 P_1 - P_2 S_2 P_2, \tag{3.11}$$

with initial conditions $P_1(0) = P_2(0) = 0$, exists on the interval $[0, L_0]$. Then the moving horizon solution exists for each horizon length $L \in [0, L_0]$. The moving horizon strategies can be written as

$$\gamma_i^{MH}(x;L) = -B_i^T P_i(L)x, \quad i = 1, 2, \quad L \in [0, L_0].$$

For the existence of a unique moving horizon solution, the regularity of $H(L)$ needs to be studied. In order to determine $H(L)$, one can either directly compute e^{ML} or solve the linear coupled initial value problem

$$H'(L) = -AH(L) + S_1G_1(L) + S_2G_2(L), \quad H(0) = I, \tag{3.12}$$

$$G_i'(L) = Q_iH(L) + A^TG_i(L), \quad G_i(0) = 0, \quad i = 1, 2. \tag{3.13}$$

Both methods do not only produce $H(L)$ but also $G_1(L)$ and $G_2(L)$, which are not interesting in determining whether the moving horizon solution exists. However, in a special situation $H(L)$ is uniquely specified by a second-order linear initial value problem, which only involves system data. This situation occurs if the matrix AS_i is symmetric for $i = 1, 2$, which is clearly a rather strong condition. Nevertheless, it is not unlikely that this condition is met in economic applications. For example, the government debt stabilization model introduced by Tabellini [115] (see also Section 3.4 of the present chapter) and the acid rain differential game of Mäler and de Zeeuw [86] can be written in such a way that AS_i is symmetric. Furthermore, this symmetry condition clearly holds in the scalar case, which is an important case for economic applications. The second-order initial value problem is formulated in the following proposition.

Proposition 3.3.3 *If the matrices AS_i are symmetric, then*

$$H''(L) = (A^2 + S_1Q_1 + S_2Q_2)H(L), \quad H(0) = I, \quad H'(0) = -A. \tag{3.14}$$

Proof Multiplying (3.13) with S_i, adding the resulting equations for $i = 1, 2$ and using the assumption $S_iA^T = AS_i$ yields

$$S_1G_1' + S_2G_2' = (S_1Q_1 + S_2Q_2)H + A(S_1G_1 + S_2G_2).$$

According to (3.12), $S_1G_1 + S_2G_2$ can be replaced by $H' + AH$, which results in

$$-AH' + S_1G_1' + S_2G_2' = (A^2 + S_1Q_1 + S_2Q_2)H.$$

Observe that the left-hand side equals H''; see (3.12). Finally, the initial conditions for H and H' follow directly from the initial conditions in (3.12) and (3.13). □

This result can of course also be used to study the existence of the open-loop Nash equilibrium for a finite-horizon time-invariant linear quadratic differential game with AS_i symmetric.

An important aspect of the moving horizon solution is stability of the corresponding closed-loop matrix. From (3.9) it follows that this matrix, which we denote by $A_{cl}^{MH}(L)$, can be written as

$$A_{cl}^{MH}(L) = A - (S_1G_1(L) + S_2G_2(L))\, H^{-1}(L). \tag{3.15}$$

In general one first needs to compute H, G_1, and G_2 for determining whether the moving horizon closed-loop matrix is stable. However, in a special situation, again met if AS_i is symmetric for $i = 1, 2$, this closed-loop matrix is uniquely specified by a first-order quadratic initial value problem, which only involves system data. This initial value problem is formulated in the next proposition.

Proposition 3.3.4 *Let $L > 0$ and assume that $H(L)$ is regular. If the matrices AS_i are symmetric, then*

$$\left(A_{cl}^{MH}\right)'(L) = \left(A_{cl}^{MH}(L)\right)^2 - \left(A^2 + S_1Q_1 + S_2Q_2\right), \quad A_{cl}^{MH}(0) = A. \tag{3.16}$$

Proof Differentiating (3.15) and using (3.12) and (3.13) yields

$$\begin{aligned}\left(A_{cl}^{MH}\right)' &= -\left(S_1G_1' + S_2G_2'\right)H^{-1} + (S_1G_1 + S_2G_2)H^{-1}H'H^{-1} = \\ &= \left(A_{cl}^{MH}\right)^2 - \left(A^2 + S_1Q_1 + S_2Q_2\right) + \left(AS_1 - S_1A^T\right)G_1H^{-1} + \\ &\quad + \left(AS_2 - S_2A^T\right)G_2H^{-1} = \left(A_{cl}^{MH}\right)^2 - \left(A^2 + S_1Q_1 + S_2Q_2\right),\end{aligned}$$

where the latter equality follows by assumption. The initial value follows from (3.12), (3.13), and (3.15). □

A problem in computing H, G_1 and G_2 might be the large number of unknowns. In the case $n = 2$, the number of unknowns in (3.12) and (3.13) already equals 12. Moreover in the N-player case the number of matrix differential equations generating H and $G_1, \ldots, G_N$, obviously equals $N + 1$. In the next proposition, this number (three if $N = 2$) is reduced to two in a special case.

Proposition 3.3.5 *Assume that $Q_i = q_iQ$ for some $q_i \in I\!R$ and some matrix Q. Let the matrix G be the solution of the inhomogeneous linear differential equation*

$$G'(L) = QH(L) + A^TG(L), \quad G(0) = 0. \tag{3.17}$$

Then $G_i(L) = q_iG(L)$ for all $L \geq 0$.

Proof Let $\tilde{G}_i(L) = q_iG(L)$, then $\tilde{G}_i'(L) = Q_iH(L) + A^T\tilde{G}_i(L)$ and $\tilde{G}_i(0) = 0$. Obviously, $\tilde{G}$ satisfies the equations by which G_i is uniquely specified, i.e. (3.13). But then it must hold that $G_i = \tilde{G}_i$. □

Remark 3.3.6 If the conditions of Propositions 3.3.3, 3.3.4, and 3.3.5 hold, the moving horizon solution can be analyzed and computed as follows. One starts by computing H from the initial value problem (3.14). From the result, the existence of the moving horizon solution can be determined. If it exists, one proceeds by computing A_{cl}^{MH} from the initial value problem (3.16). At this point in the analysis, it can be concluded whether the moving horizon solution is stabilizing. Given the closed-loop matrix A_{cl}^{MH}, (3.15) can be rewritten as $GH^{-1} = (q_1S_1 + q_2S_2)^{-1}(A - A_{cl})$, provided that $q_1S_1 + q_2S_2$ is regular. Consequently, (3.9) produces the following expression for the moving horizon solution:

$$\gamma_i^{MH}(x; L) = -q_iB_i^T(q_1S_1 + q_2S_2)^{-1}(A - A_{cl}^{MH}(L))x, \tag{3.18}$$

provided that $q_1S_1 + q_2S_2$ is regular. □

This subsection ends with the following theorem, which deals with the behavior of the moving horizon solution as the horizon length L tends to infinity. Clearly, this behavior is closely connected to the limiting behavior of the open-loop Nash equilibrium of the finite horizon differential game as the planning horizon tends to infinity.

Theorem 3.3.7 *Consider the differential game and the notations introduced in Theorem 3.3.1. Assume that $H(L)$ is regular for all $L > 0$. Define $P_i := G_iH^{-1}$ and assume that $\overline{P}_i := \lim_{L\to\infty} P_i(L)$ exists for $i = 1, 2$. Furthermore, assume that $A - S_1\overline{P}_1 - S_2\overline{P}_2$ is stable. Then the open-loop control functions corresponding to the moving horizon strategies $\gamma_i^{MH}(x; L)$ that are obtained in the limit $L \to \infty$ constitute an open-loop Nash equilibrium for the infinite-horizon differential game under consideration.*

Proof According to Corollary 3.3.2, $P_1(L)$ and $P_2(L)$ satisfy the set of open-loop Riccati differential equations (3.10) and (3.11). Since the limit of $P_i(L)$ for $L \to \infty$ exists, $\overline{P}_1$ and $\overline{P}_2$ satisfy the corresponding set of open-loop algebraic Riccati equations. It follows from [45, Theorem 12] that under the additional stability condition on $A - S_1\overline{P}_1 - S_2\overline{P}_2$, the control functions

$$u_i(t) = -B_i^T\overline{P}_ie^{t(A-S_1\overline{P}_1-S_2\overline{P}_2)}x_0, \quad i = 1, 2,$$

where the initial state is denoted by x_0, constitute an open-loop Nash equilibrium for the infinite-horizon game. The theorem now follows from the observation that these open-loop equilibrium control functions can in feedback form be written as $-B_i^T\overline{P}_ix = \lim_{L\to\infty} \gamma_i^{MH}(x; L)$. □

A more detailed analysis of the limiting behavior of the open-loop Nash equilibrium for the finite horizon game as the planning horizon tends to infinity can be found in [10, Section 6.5.3]. For example, relatively weak sufficient conditions on the system data for the conditions listed in Theorem 3.3.7 to hold are given in this reference.

3.3.2 Scalar Case

The formulation of a linear-quadratic differential game simplifies substantially if the system parameters are all scalar. In this section it is assumed that this is the case and in order to indicate the difference with the general formulation of the previous section, the system parameters A, B_i, and Q_i are denoted by a, b_i, and q_i, respectively. Recall that it is assumed that $q_i \geq 0$. The conditions on the system data listed in the Propositions 3.3.3, 3.3.4, and 3.3.5 are all satisfied. For that reason, the scalar case can conveniently be analyzed using the lines of Remark 3.3.6.

Let

$$\mu := \sqrt{a^2 + s_1 q_1 + s_2 q_2}, \quad s_i := b_i^2, \tag{3.19}$$

then, according to Proposition 3.3.3, H satisfies $H'' - \mu^2 H = 0$, $H(0) = 1$ and $H'(0) = -a$. Consequently, $H(L) = \cosh(\mu L) - (a/\mu)\sinh(\mu L)$ if $\mu > 0$ and $H(L) = 1$ if $\mu = 0$. In both cases we have $H(L) \neq 0$ for all $L > 0$. Hence, Theorem 3.3.1 shows that the moving horizon solution exists for all $L > 0$.

Denote the moving horizon closed-loop matrix (a scalar in the present case) by $a_{MH}(L)$. Then, according to Proposition 3.3.4, a_{MH} satisfies the quadratic differential equation $a'_{MH} = a^2_{MH} - \mu^2$ with initial condition $a_{MH}(0) = a$. The solution can be obtained by separating the variables. We have $da_{MH}/(a^2_{MH} - \mu^2) = dL$. Since

$$\int \frac{da_{MH}}{a^2_{MH} - \mu^2} = \frac{i}{\mu} \arctan \frac{iL}{\mu}$$

for $\mu \neq 0$, it follows that $\arctan ia_{MH}/\mu = -i\mu L + C_0$ for some constant C_0, or equivalently, $a_{MH} = -\mu \tanh(\mu L + C)$ for some constant C. For reference below, we state this result as a separate lemma.

Lemma 3.3.8 *Let σ be a nonzero real number. The solution of the differential equation $x' = x^2 - \sigma^2$ is given by $x(L) = -\sigma \tanh(\sigma L + C)$ with C an arbitrary constant.*

The constant C can be determined using the initial condition $a_{MH}(0) = a$. We obtain

$$a_{MH}(L) = -\mu \tanh(\mu L - \lambda), \quad \lambda := \frac{1}{2} \log \frac{\mu + a}{\mu - a}, \tag{3.20}$$

if $\mu \neq 0$, or equivalently, if $s_1 q_1 + s_2 q_2 > 0$. If $s_1 q_1 + s_2 q_2 = 0$, then $a_{MH}(L) = a$. This degenerate case will not be further analyzed, i.e. we assume that $s_1 q_1 + s_2 q_2 > 0$. From (3.20), we see that a_{MH} is strictly decreasing and approaches $-\mu$ as L approaches infinity. Furthermore, a_{MH} has a unique zero at $\overline{L} := \lambda/\mu$. Hence, the stability of the moving horizon solution depends on the sign of $\overline{L}$. Indeed, if $\overline{L} \leq 0$, the moving horizon solution is stabilizing for all $L > 0$, and if $\overline{L} > 0$ the stabilizing property holds if and only if $L > \overline{L}$. It is easily seen that the sign of $\overline{L}$ is uniquely specified by the sign of a. The results obtained so far are summarized in the following theorem.

Theorem 3.3.9 *Consider the two-player infinite-horizon linear-quadratic time-invariant differential game*

$$\dot{x} = ax + b_1u_1 + b_2u_2, \quad J_i = \int_0^\infty \left(q_i x^2 + u_i^2\right) dt$$

with $q_i \geq 0$. *There exists a unique moving horizon solution for all horizon lengths* L. *Further, let the numbers* s_i *and* μ *be defined by* (3.19), *assume that* $s_1q_1 + s_2q_2 > 0$, *and define*

$$\overline{L} := \frac{1}{2\mu} \log \frac{\mu + a}{\mu - a}.$$

Then if $a \leq 0$ *the moving horizon closed-loop system is stable for all* $L > 0$ *and if* $a > 0$ *this closed-loop system is unstable for* $0 < L \leq \overline{L}$ *and stable for* $L > \overline{L}$.

Remark 3.3.10 If $a > 0$, then

$$\overline{L} = \frac{1}{2a\sqrt{1 + (s_1q_1 + s_2q_2)/a^2}} \log \frac{\sqrt{1 + (s_1q_1 + s_2q_2)/a^2} + 1}{\sqrt{1 + (s_1q_1 + s_2q_2)/a^2} - 1}.$$

This expression shows that if $(s_1q_1 + s_2q_2)/a^2$ is close to zero, then $\overline{L}$ is relatively large. This is for instance the case if $s_1q_1 + s_2q_2 << a^2$. In this case relatively large horizon lengths are required to stabilize the closed-loop system. Alternatively, if $(s_1q_1 + s_2q_2)/a^2$ is large, which happens for instance if $s_1q_1 + s_2q_2 >> a^2$, then $\overline{L}$ is close to zero. In this case, also small horizon lengths are stabilizing. □

The monotonicity property of a_{MH} indicates that for increasing values of L, the convergence speed of the closed-loop moving horizon dynamics also increases. Indeed, for a small stabilizing value of the planning horizon length L, the players continuously consider it important to minimize short run costs. By implementing the corresponding control actions it is to be expected that it lasts a relatively long time before the system reaches its steady state value. This phenomenon

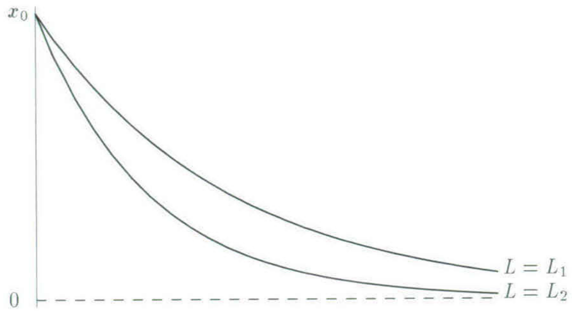


Figure 3.1: Moving Horizon State Trajectories ($L_1 < L_2$).

is illustrated by Figure 3.1, where the graphs of two stable moving horizon state trajectories are drawn for the same positive initial state.

The moving horizon strategies follow immediately from (3.18). We have

$$\gamma_i^{MH}(x; L) = -\frac{b_i q_i (a + \mu \tanh(\mu L - \lambda))}{s_1 q_1 + s_2 q_2} x, \quad i = 1, 2. \tag{3.21}$$

For each initial state x_0, a unique open-loop control function corresponds to (3.21) for each player. We denote this control function by $u_i^{MH}(t; x_0, L)$. The behavior of $u_i^{MH}(t; x_0, L)$ is characterized by the following theorem and illustrated by Figure 3.2.

Theorem 3.3.11 *Consider the differential game and notations introduced in Theorem 3.3.9. Assume that* $s_1 q_1 + s_2 q_2 > 0$. *The moving horizon strategies* $\gamma_i^{MH}(x; L)$ *are given by* (3.21). *Denote the corresponding control functions by* $u_i^{MH}(t; x_0, L)$, *where* x_0 *is the initial state. Then, the trajectory* $u_i^{MH}(t; x_0, L)$ *is strictly increasing if* $b_i x_0 > 0$ *and strictly decreasing if* $b_i x_0 < 0$ *for all* $L > \max\{0, \overline{L}\}$. *For any horizon lengths* L_1 *and* L_2 *with* $\max\{0, \overline{L}\} < L_1 < L_2$ *we have*

$$0 < u_i^{MH}(0; x_0, L_1) < u_i^{MH}(0; x_0, L_2) < \lim_{L \to \infty} u_i^{MH}(0; x_0, L) = -\frac{b_i q_i x_0}{\mu - a}, \tag{3.22}$$

if $b_i x_0 < 0$, *and*

$$-\frac{b_i q_i x_0}{\mu - a} = \lim_{L \to \infty} u_i^{MH}(0; x_0, L) < u_i^{MH}(0; x_0, L_2) < u_i^{MH}(0; x_0, L_1) < 0, \tag{3.23}$$

if $b_i x_0 > 0$. *Finally, there is a unique number* $t > 0$ *such that* $u_i^{MH}(t; x_0, L_1) = u_i^{MH}(t; x_0, L_2)$.

Proof We have

$$u_i^{MH}(t; x_0, L) = -\frac{b_i q_i x_0}{q_1 s_1 + q_2 s_2}(a - a_{MH}(L))e^{a_{MH}(L)t}.$$

The first part is readily seen by computing u_i^{MH}. The second part ((3.22) and (3.23)) follows from the monotonicity of a_{MH} and the fact that $a_{MH}(L) \to -\mu$ as $L \to \infty$. For the third part, note that the equation $u_i^{MH}(t; x_0, L_1) = u_i^{MH}(t; x_0, L_2)$ is equivalent to

$$e^{(a_{MH}(L_1) - a_{MH}(L_2))t} = \frac{a - a_{MH}(L_2)}{a - a_{MH}(L_1)}.$$

Since a_{MH} is strictly decreasing, this equation has exactly one solution $t > 0$. □

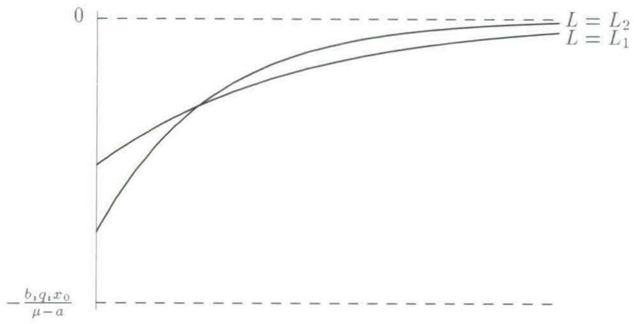


Figure 3.2: Moving Horizon Control Trajectories ($L_1 < L_2$).

Two graphs of $u_i^{MH}(t; x_0, L)$ in the case $b_i x_0 > 0$ are drawn in Figure 3.2. They correspond to two different values of the horizon length L. Let t^* be the t-coordinate corresponding to the intersection point of the two graphs. By increasing L from L_1 to L_2 the initial moving horizon control action decreases. In fact it is readily seen from the figure that this property holds for all t between 0 and t^*. For $t > t^*$ the situation is the other way around, i.e. by increasing L from L_1 to L_2 the control value increases. Hence, for small values of the planning horizon length the short run moving horizon control actions, i.e. for t between 0 and t^* are closer to the target value zero than for large values of the planning horizon length. In the long run, i.e. for $t > t^*$, the moving horizon control actions are closer to the target value for larger values of the planning horizon length. This property is clearly consistent with the moving horizon concept. By implementing the moving horizon solution for a small value of L, the players gain in the short run a control action relatively close to the target value. However, they pay a price in the long run, where their control action is relatively far from the target value.

In both Figures 3.1 and 3.2, one can easily compare the moving horizon solution trajectories for some initial state with the open-loop Nash equilibrium on the infinite horizon corresponding to the same initial state. Indeed, Theorem 3.3.7 implies that for large values of L_2, the corresponding state and control trajectories resemble the optimal trajectories from the open-loop Nash equilibrium on the infinite horizon. It is clearly visible that the convergence speed of all the trajectories is higher for the open-loop Nash equilibrium than for the moving horizon solution.

3.3.3 An Endogenous Moving Horizon Length

The horizon length L could be considered as an exogenous parameter. It might however be desirable to avoid the choice of a moving horizon length. The present section suggests a method to make L an endogenous parameter. Since the moving horizon solution fits in the scope of differential games with feedback information structure, it seems plausible to minimize the distance between a feedback Nash equilibrium and the moving horizon solution by varying L. The analysis in this section is limited to the scalar case and ends with a remark about how one could exploit the minimization process in the general case.

It is well-known [10, Section 6.5.3] that feedback Nash equilibria for an infinite-horizon linear-quadratic differential game are generated by solutions (k_1, k_2) of the set of feedback algebraic Riccati equations

$$q_1 + 2ak_1 - s_1k_1^2 - 2s_2k_1k_2 = 0,$$
$$q_2 + 2ak_2 - 2s_1k_1k_2 - s_2k_2^2 = 0.$$

If a pair of non-negative numbers (k_1, k_2) satisfies this set of equations, then the feedback strategies

$$\gamma_i^{FB}(x) = -b_ik_ix, \quad i = 1, 2,$$

constitute a feedback Nash equilibrium. We measure the distance[1] between a feedback Nash equilibrium and the moving horizon solution by the Euclidean distance between the corresponding pair of feedback maps, or equivalently, between the points (see (3.21))

$$(k_1, k_2) \quad \text{and} \quad \frac{a + \mu \tanh(\mu L - \lambda)}{s_1q_1 + s_2q_2}(q_1, q_2).$$

[1]Other choices could have been made for a distance measure; for instance in terms of the corresponding cost values.

It is convenient to introduce the notations $\sigma_i := s_i q_i$ and $\kappa_i := s_i k_i$. The distance between a feedback Nash equilibrium and the moving horizon solution is then measured by the distance between the points

$$(\kappa_1, \kappa_2) \quad \text{and} \quad \frac{a + \mu \tanh(\mu L - \lambda)}{\sigma_1 + \sigma_2}(\sigma_1, \sigma_2),$$

where (κ_1, κ_2) is a pair of non-negative numbers satisfying the set of equations

$$\sigma_1 + 2a\kappa_1 - \kappa_1^2 - 2\kappa_1\kappa_2 = 0, \tag{3.24}$$

$$\sigma_2 + 2a\kappa_2 - 2\kappa_1\kappa_2 - \kappa_2^2 = 0. \tag{3.25}$$

The graphs of the equations in this set are hyperbolas in the (κ_1, κ_2)-plane (these equations are studied in more detail in Section 5.6 of the present thesis). In [122] it has been shown that this set of feedback algebraic Riccati equations has either one or three solutions in the first quadrant of the (κ_1, κ_2)-plane. This has been further specified in [46, Theorem 2]: if $a \leq 0$, there exists one such solution and if $a > 0$, there exists a unique solution if and only if

$$3(\mu^4 - 3\sigma_1\sigma_2)^2 - 4a^2\mu^6 > 0. \tag{3.26}$$

If $a > 0$ and (3.26) does not hold, there are three solutions. One can easily verify that the graph corresponding to (3.24) approaches infinity for $\kappa_1 \downarrow 0$, strictly decreases for $\kappa_1 > 0$, and intersects the κ_1-axis at $\kappa_1 = a + \sqrt{a^2 + \sigma_1}$. Similarly, the graph corresponding to (3.25) intersects the κ_2-axis at $\kappa_2 = a + \sqrt{a^2 + \sigma_2}$, strictly decreases for $\kappa_1 > 0$, and approaches zero for $\kappa_1 \to \infty$; see also Figure 3.3.

Figure 3.3: Sketch of Hyperbolas with One and Three Intersection Points

The moving horizon solution corresponds uniquely to the point

$$(\kappa_1^{MH}(L), \kappa_2^{MH}(L)) = \kappa^{MH}(L)(\sigma_1, \sigma_2), \quad \kappa^{MH}(L) := \frac{a + \mu \tanh(\mu L - \lambda)}{\sigma_1 + \sigma_2}. \tag{3.27}$$

The graph $\left\{(\kappa_1^{MH}(L), \kappa_2^{MH}(L)| \max\{0, \overline{L}\} < L < \infty\right\}$ is a line segment from $(0, 0)$ if $a \leq 0$, or from $\frac{a}{\sigma_1+\sigma_2}(\sigma_1, \sigma_2)$ if $a > 0$, to $\frac{a+\mu}{\sigma_1+\sigma_2}(\sigma_1, \sigma_2)$. This line segment consists of all stabilizing moving horizon solutions; we refer to it as the moving horizon graph.

The geometrical structure of both the feedback Nash equilibrium and the moving horizon solution concept allows for a convenient numerical determination of the moving horizon length L that minimizes the distance between a feedback Nash equilibrium and the moving horizon solution. If there exists a unique feedback Nash equilibrium, this minimization produces a unique, endogenously determined moving horizon length. In case of three such equilibria, three horizon lengths result, and one could take the horizon length corresponding to the smallest distance. Additionally, this method selects generically one of the three equilibria with the property that it is closest (with respect to the distance measure chosen here) to the moving horizon solution for some specific unique horizon length. The numerical procedure is illustrated by the following example.

Example 3.3.12 Let $a = 3$, $\sigma_1 = 2$, and $\sigma_2 = 1$. Then $\mu = 2\sqrt{3}$. One can easily verify that (3.26) is violated, implying that there are three feedback Nash equilibria. The two hyperbolas in the (κ_1, κ_2)-plane corresponding to the equations (3.24) and (3.25) are drawn in Figure 3.4. This

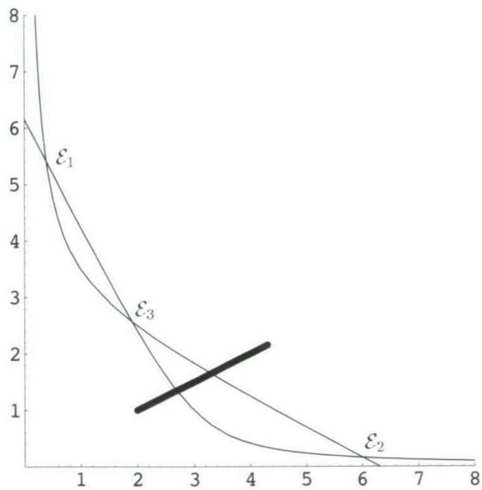


Figure 3.4: Hyperbolas and Moving horizon Graph Corresponding to Example 3.3.12.

picture clearly shows the existence of three feedback Nash equilibria. The corresponding intersection points are denoted by $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$. The thick line segment in this picture is the moving horizon graph. It is clear from inspection that the distance between equilibrium $\mathcal{E}_2$ and the moving horizon graph is minimal at the end-point of the moving horizon graph. This corresponds to a horizon length $L = \infty$. The minimal distance between the other two equilibria and the moving horizon graph can be found as follows. The line perpendicular to the moving horizon graph and intersecting the κ_2-axis at $(0, \alpha)$ is described by the equation

$$\kappa_2 = -\frac{\sigma_1}{\sigma_2}\kappa_1 + \alpha, \quad \alpha \in [\alpha_1, \alpha_2], \tag{3.28}$$

where

$$\alpha_1 := \frac{a(\sigma_1^2 + \sigma_2^2)}{\sigma_2(\sigma_1 + \sigma_2)}, \quad \alpha_2 := \frac{(a + \mu)(\sigma_1^2 + \sigma_2^2)}{\sigma_2(\sigma_1 + \sigma_2)}.$$

Apparently, there exist two solutions $(\kappa_1, \kappa_2, \alpha)$, with $\kappa_i > 0$ and $\alpha \in [\alpha_1, \alpha_2]$, of the set of equations (3.24), (3.25), and (3.28). The pair (κ_1, κ_2) of such a solution corresponds to either equilibrium $\mathcal{E}_1$ or $\mathcal{E}_3$. The number α in this solution can be used to find the point at the moving horizon graph which corresponds to the minimal distance of the moving horizon graph and the corresponding equilibrium. Finally, one finds the corresponding moving horizon length L from (3.27). It is clear from inspection that equilibrium $\mathcal{E}_3$ has the smallest distance to the moving horizon graph and one can compute the corresponding horizon length $L \approx 0.45$. This number is close to $\overline{L} \approx 0.38$. □

Remark 3.3.13 The idea of making the moving horizon length endogenous by minimizing a distance between the moving horizon length and a feedback Nash equilibrium could be generalized to the multi-variable case as follows. Assume that the solutions P_1^{OL} and P_2^{OL} of the set of open-loop Riccati differential equations (3.10) and (3.11) with initial condition $P_1^{OL}(0) = P_2^{OL}(0) = 0$, exist. Define

$$\begin{aligned}
\varphi_1 &:= Q_1 + A^T P_1^{OL} + P_1^{OL} A - P_1^{OL} S_1 P_1^{OL} - P_1^{OL} S_2 P_2^{OL} - P_2^{OL} S_2 P_1^{OL}, \\
\varphi_2 &:= Q_2 + A^T P_2^{OL} + P_2^{OL} A - P_2^{OL} S_1 P_1^{OL} - P_2^{OL} S_2 P_2^{OL} - P_1^{OL} S_1 P_2^{OL}.
\end{aligned}$$

Here, the right-hand sides correspond to the set of feedback algebraic Riccati equations. Now, one could select a moving horizon length L such as to minimize the expression $\|\varphi_1(L)\| + \|\varphi_2(L)\|$, where $\|\cdot\|$ is a matrix norm. Clearly, the resulting length depends on the matrix norm that has been chosen. □

3.4 A Government Debt Stabilization Game

The present section illustrates the analytic computability of the moving horizon solution in an economic example where feedback Nash equilibria cannot be determined analytically. This economic example is taken from [2]; it has originally been introduced by Tabellini [115]. It is a linear-quadratic differential game, thus the theory of the previous section is applicable. The model concerns monetary and fiscal policy interaction in a government debt stabilization differential game. In this game, the government debt is modeled as the state and the players are fiscal and monetary authorities. The differential equation is given by

$$\dot{d} = rd + f - m, \quad d(0) = d_0,$$

with d the government debt, f the primary fiscal deficits controlled by the fiscal authority, and m the seignorage controlled by the monetary authority. Furthermore, rd represents the interest payments on government debt and d_0 the initial stock of outstanding government debt. The objectives of the players are described by the cost functionals

$$J_f(f,m) = \int_0^\infty \left((f-\overline{f})^2 + \kappa_1(d-\overline{d})^2\right)\exp(-\delta t)dt$$

and

$$J_m(f,m) = \int_0^\infty \left((m-\overline{m})^2 + \kappa_2(d-\overline{d})^2\right)\exp(-\delta t)dt,$$

respectively. The parameters κ_1 and κ_2 are weights, $\overline{f}$, $\overline{m}$, and $\overline{d}$ are given target values, and δ is a discounting factor. All the parameters introduced so far, i.e. r, d_0, the weights, the targets and the discounting factor are assumed to be positive. Furthermore, it is assumed that $d_0 - \overline{d} > 0$, $r\overline{d}+\overline{f}-\overline{m} > 0$, and $\delta - r \geq 0$. These assumptions are taken from [2, Section 2], in which one can find a detailed description of the differential game under consideration. They computed the open-loop Nash equilibrium and the cooperative solution. The open-loop Nash equilibrium has been computed in a different manner by Engwerda [45]. The feedback Nash equilibrium seems not analytically computable. In what follows, the moving horizon solution of the government-debt stabilization game will be computed and analyzed following the lines of Remark 3.6.

The model under consideration can be written in the linear-quadratic format (3.4), (3.5), and (3.6), by introducing a two-dimensional state variable $x = [x_1 \;\, x_2]^T$, and two scalar control variables

u_1 and u_2 as follows (see also [45, Section 6]):

$$x_1 = (d - \overline{d})\exp(-\delta t/2); \tag{3.29}$$

$$x_2 = (r\overline{d} + \overline{f} - \overline{m})\exp(-\delta t/2); \tag{3.30}$$

$$u_1 = (f - \overline{f})\exp(-\delta t/2); \tag{3.31}$$

$$u_2 = (m - \overline{m})\exp(-\delta t/2). \tag{3.32}$$

Furthermore, the matrices A, B_i, and Q_i are defined by

$$A := \begin{bmatrix} r - \frac{\delta}{2} & 1 \\ 0 & -\frac{\delta}{2} \end{bmatrix}, \quad B_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad Q_i := \kappa_i \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.33}$$

Since

$$S_1 = S_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

it follows that AS_i is symmetric. Hence, according to Proposition 3.3, it follows that H satisfies the second order differential equation (3.14). Write $H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$, then, since

$$A^2 + S_1Q_1 + S_2Q_2 = \begin{bmatrix} \alpha^2 & r - \delta \\ 0 & \delta^2/4 \end{bmatrix},$$

with

$$\alpha := \sqrt{(r - \delta/2)^2 + \kappa}, \quad \kappa := \kappa_1 + \kappa_2, \tag{3.34}$$

the matrix elements of H satisfy the following coupled system of second-order linear differential equations:

$$h''_{11} = \alpha^2 h_{11} + (r - \delta)h_{21}, \quad h_{11}(0) = 1, \quad h'_{11}(0) = -(r - \delta/2); \tag{3.35}$$

$$h''_{12} = \alpha^2 h_{12} + (r - \delta)h_{22}, \quad h_{12}(0) = 0, \quad h'_{12}(0) = -1; \tag{3.36}$$

$$h''_{21} = (\delta^2/4)h_{21}, \quad h_{21}(0) = 0, \quad h'_{21}(0) = 0; \tag{3.37}$$

$$h''_{22} = (\delta^2/4)h_{22}, \quad h_{22}(0) = 1, \quad h'_{22}(0) = \delta/2. \tag{3.38}$$

The last two equations are in fact decoupled equations. The solutions are easily found. Indeed, we have $h_{21}(L) = 0$ and $h_{22}(L) = \exp(\delta L/2)$. This reduces (3.35) to $h''_{11} = \alpha^2 h_{11}$ and thus $h_{11}(L) = \cosh(\alpha L) - ((r - \delta/2)/\alpha)\sinh(\alpha L)$. Hence

$$\det H(L) = \left(\cosh(\alpha L) - \frac{r - \delta/2}{\alpha}\sinh(\alpha L)\right)\exp(\delta L/2). \tag{3.39}$$

Since $|r - \delta/2| < \alpha$, it follows that $\det H(L) \neq 0$ for all $L > 0$, and hence, the moving horizon solution exists for all $L > 0$. According to Proposition 3.4, the closed-loop matrix is uniquely specified by the initial value problem (3.16). Write $A_{cl}^{MH} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, then $a_1, \ldots, a_4$ satisfy the following coupled system of first-order quadratic differential equations:

$$a_1' = a_1^2 + a_2 a_3 - \alpha^2, \qquad a_1(0) = r - \delta/2, \tag{3.40}$$

$$a_2' = a_2(a_1 + a_4) - (r - \delta), \quad a_2(0) = 1, \tag{3.41}$$

$$a_3' = a_3(a_1 + a_4), \qquad a_3(0) = 0, \tag{3.42}$$

$$a_4' = a_2 a_3 + a_4^2 - \delta^2/4, \qquad a_4(0) = -\delta/2. \tag{3.43}$$

It is immediately clear that $a_3(L) = 0$, which reduces (3.40) and (3.43) to the decoupled initial value problems

$$a_1' = a_1^2 - \alpha^2, \qquad a_1(0) = r - \delta/2; \tag{3.44}$$

$$a_4' = a_4^2 - \delta^2/4, \quad a_4(0) = -\delta/2. \tag{3.45}$$

The solutions can be determined using Lemma 3.3.8. Indeed, this lemma shows that the solution of (3.44) is given by

$$a_1(L) = -\alpha \tanh\left(\alpha L - \theta\right), \tag{3.46}$$

with

$$\theta := -\tanh^{-1} \frac{r - \delta/2}{-\alpha} = \frac{1}{2} \log \frac{\alpha + r - \delta/2}{\alpha - r + \delta/2}. \tag{3.47}$$

Due to the fact that $|r - \delta/2| < \alpha$, the argument of the logarithm is positive, which implies that the number θ is well defined. The solution of (3.45) can also be determined from Lemma 3.3.8. However, since the initial condition is an equilibrium point of the differential equation, we immediately deduce that the solution is constant, i.e.

$$a_4(L) = -\frac{\delta}{2}. \tag{3.48}$$

Substituting (3.46) and (3.48) in (3.41) yields the following inhomogeneous linear initial value problem for the function a_2:

$$a_2' = -\left(\frac{\delta}{2} + \alpha \tanh(\alpha L - \theta)\right) a_2 - (r - \delta), \quad a_2(0) = 1. \tag{3.49}$$

An inhomogeneous quasi-linear differential equation $x' = \rho x + \sigma$ can be solved in the following way. Substituting $x = x_1 x_2$ yields $x_1' x_2 + x_1 x_2' = \rho x_1 x_2 + \sigma$, or equivalently, $x_2(x_1' - \rho x_1) = \sigma - x_1 x_2'$. Now, one first solves the homogeneous equation $x_1' = \rho x_1$ by separating the variables. Next, x_2 can be determined from $x_2' = \sigma / x_1$, which produces $x = x_1 x_2$. Thus in order to solve (3.49) we set $a_2 = vw$ and consider the homogeneous equation

$$v' = -\left(\frac{\delta}{2} + \alpha \tanh(\alpha L - \theta)\right) v.$$

Separating the variables and integrating yields (we can omit the integration constant here)

$$\log v = -\int \left(\frac{\delta}{2} + \alpha \tanh(\alpha L - \theta)\right) dL = -\frac{\delta L}{2} - \log\cosh(\alpha L - \theta).$$

Thus w needs to be determined from $w' = -(r - \delta) e^{\delta L/2} \cosh(\alpha L - \theta)$. By integrating this equation it follows that

$$\begin{aligned} w &= -(r-\delta) \int e^{\delta L/2} \cosh(\alpha L - \theta) dL = \\ &= \frac{(r-\delta)}{\beta} e^{\delta L/2} \left(\frac{\delta}{2} \cosh(\alpha L - \theta) - \alpha \sinh(\alpha L - \theta)\right) + C, \end{aligned}$$

with C an arbitrary constant and where we introduced the constant

$$\beta := \kappa_1 + \kappa_2 - r(\delta - r).$$

Hence we obtain

$$a_2(L) = \frac{r-\delta}{\beta}\left(\frac{\delta}{2} - \alpha \tanh(\alpha L - \theta)\right) + C \frac{e^{-\delta L/2}}{\cosh(\alpha L - \theta)}.$$

The constant C can be determined from the initial condition $a_2(0) = 1$. For this we shall use the properties

$$\cosh\theta = \frac{\alpha}{\sqrt{\kappa}}, \quad \sinh\theta = \frac{r - \delta/2}{\sqrt{\kappa}},$$

which can easily be seen using the identities

$$\cosh\log\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{1}{2}} = \frac{1}{\sqrt{1-\sigma^2}}, \quad \sinh\log\left(\frac{1+\sigma}{1-\sigma}\right)^{\frac{1}{2}} = \frac{\sigma}{\sqrt{1-\sigma^2}}, \quad |\sigma| < 1.$$

By applying these properties we straightforwardly derive that $C = \alpha\sqrt{\kappa}/\beta$, and thus the solution of (3.49) is given by

$$a_2(L) = \frac{1}{\beta}\left((r-\delta)\left(\frac{\delta}{2} - \alpha \tanh(\alpha L - \theta)\right) + \frac{\alpha\sqrt{\kappa} e^{-\delta L/2}}{\cosh(\alpha L - \theta)}\right). \tag{3.50}$$

Using the closed-loop matrix A_{cl}^{MH}, we next determine the moving horizon government-debt trajectory, which we denote by $d^{MH}(t;L)$. From this trajectory we shall study among other things the moving horizon closed-loop stability. The moving horizon state variables can be determined from the closed-loop dynamics

$$\dot{x}_1 = a_1(L)x_1 + a_2(L)x_2, \qquad x_1(0) = x_{10} := d_0 - \overline{d}; \tag{3.51}$$

$$\dot{x}_2 = a_4(L)x_2, \qquad x_2(0) = x_{20} := r\overline{d} + \overline{f} - \overline{m}. \tag{3.52}$$

This initial value problem can be solved as follows. From (3.52) it is immediately clear that

$$x_2 = e^{a_4 t}x_{20}. \tag{3.53}$$

Substituting this in (3.51) results in a scalar linear inhomogeneous differential equation for the variable x_1. Substituting in this equation $x_1 = \xi_1\xi_2$ shows that if ξ_1 is a solution of the corresponding homogeneous equation, ξ_2 can be determined from $\xi_1\dot{\xi}_2 = a_2e^{a_4t}x_{20}$. It is easily seen that ξ_1 can be chosen as $\xi_1 = e^{a_1t}$. Hence, ξ_2 satisfies $\dot{\xi}_2 = a_2e^{(a_4-a_1)t}x_{20}$, which shows that we have to distinguish between the cases $a_1 = a_4$ and $a_1 \neq a_4$. We obtain

$$x_1 = \begin{cases} (a_2x_{20}t + C)e^{a_1t}, & a_1 = a_4; \\ \\ \dfrac{a_2x_{20}}{a_4 - a_1}e^{a_4t} + Ce^{a_1t}, & a_1 \neq a_4. \end{cases}$$

Here C is an arbitrary constant which can be determined using the initial condition $x_1(0) = x_{10}$. This produces

$$x_1 = \begin{cases} (a_2x_{20}t + x_{10})e^{a_1t}, & a_1 = a_4; \\ \\ \dfrac{a_2x_{20}}{a_4 - a_1}(e^{a_4t} - e^{a_1t}) + x_{10}e^{a_1t}, & a_1 \neq a_4. \end{cases} \tag{3.54}$$

Using this result and (3.29) we obtain

$$d^{MH}(t;L) = \begin{cases} (a_2(L)x_{20}t + x_{10})e^{-h(L)t}, & a_1(L) = a_4(L); \\ \\ d_\infty(L) + (d_0 - d_\infty(L))e^{-h(L)t}, & a_1(L) \neq a_4(L). \end{cases} \tag{3.55}$$

with

$$h(L) := \alpha\tanh(\alpha L - \theta) - \delta/2, \tag{3.56}$$

$$d_\infty(L) := \overline{d} + \frac{x_{20}}{\beta}(\delta - r + \alpha\sqrt{\kappa}v(L)), \tag{3.57}$$

where

$$v(L) := \frac{e^{-\delta L/2}}{\alpha \sinh(\alpha L - \theta) - \frac{\delta}{2}\cosh(\alpha L - \theta)}.$$

Note that $a_1(L) = a_4(L)$ if and only if $h(L) = 0$. The function h is called the adjustment speed. The government-debt dynamics is stable, i.e. it reaches a steady-state value, if and only if $h(L)$ is positive. Clearly, if $\alpha - \delta/2 < 0$, $h(L)$ is negative for all $L > 0$. If $\alpha - \delta/2 > 0$, there exists a number $\hat{L}$, i.e. the unique root of $h(L) = 0$, such that $h(L) < 0$ for $L < \hat{L}$, and $h(L) > 0$ for $L > \hat{L}$. The condition $\alpha - \delta/2 > 0$ is equivalent to the condition $\beta > 0$. In the analysis of van Aarle et al. [2] the same condition turned out to be necessary and sufficient for the open-loop Nash equilibrium to be stable. Indeed, according to Theorem 3.7, if $L \to \infty$, the open-loop controls corresponding to the moving horizon solution constitute the open-loop Nash equilibrium. In the sequel it is assumed that $\beta > 0$. Next we determine an explicit expression for the critical value $\hat{L}$. The equation $h(L) = 0$ can be written as $\tanh(\alpha L - \theta) = \delta/2\alpha$. Thus

$$\alpha \hat{L} - \theta = \tanh^{-1}\frac{\delta}{2\alpha} = \frac{1}{2}\log\frac{\alpha + \delta/2}{\alpha - \delta/2},$$

or equivalently,

$$\hat{L} = \frac{1}{2\alpha}\log\frac{(\alpha + r - \delta/2)(\alpha + \delta/2)}{(\alpha - (r - \delta/2))(\alpha - \delta/2)}. \tag{3.58}$$

Lemma 3.4.1 *The critical value $\hat{L}$ is positive.*

Proof The argument of the logarithm in (3.58) can be rewritten as

$$\frac{(1 + (r - \delta/2)/\alpha)(1 + \delta/(2\alpha))}{(1 - (r - \delta/2)/\alpha)(1 - \delta/(2\alpha)}.$$

Recall that $|r - \delta/2| < \alpha$. Furthermore, since $\beta > 0$, we have $\alpha > \delta/2$. Consider the function $\psi(x, y) = \frac{(1+x)(1+y)}{(1-x)(1-y)}$ for $|x| < 1$ and $0 < y < 1$. It is easily seen that $\psi(x, y) > 1$ if and only if $y > -x$. Clearly we have $\delta/(2\alpha) > (r - \delta/2)/\alpha$, which completes the proof. □

This lemma implies that if the moving horizon length is small, i.e. smaller than $\hat{L}$, the moving horizon government debt converges to infinity exponentially fast. If $L = \hat{L}$ we see from (3.55) that the moving horizon government debt converges to infinity linearly. Let us further consider the case that d^{MH} reaches a steady state value as time goes to infinity. Recall from (3.55) that

$$d^{MH}(t; L) = d_\infty(L) + (d_0 - d_\infty(L))e^{-h(L)t}, \quad L > \hat{L}. \tag{3.59}$$

Note that $d^{MH}(t;L) \to d_\infty(L)$ as $t \to \infty$ for all $L > \hat{L}$, justifying the notation d_∞ for the steady state debt. In the next lemma the steady state debt is investigated in more detail; specifically in relation to the initial debt as a function of L.

Lemma 3.4.2 *If $x_{10}-(\delta-r)x_{20}/\beta \leq 0$, then $d_\infty(L) > d_0$ for all $L > \hat{L}$. If $x_{10}-(\delta-r)x_{20}/\beta > 0$, then there exists an $L^* > \hat{L}$ so that $d_\infty(L) > d_0$ for all $\hat{L} < L < L^*$, $d_\infty(L^*) = d_0$ and $d_\infty(L) < d_0$ for all $L > L^*$.*

Proof Because $h(L) > 0$ for all $L > \hat{L}$, also $v(L) > 0$ for all $L > \hat{L}$. Moreover, it is easily seen that $v(L) \to \infty$ as $L \downarrow \hat{L}$, $v(L) \downarrow 0$ as $L \to \infty$, and v is strictly decreasing for all $L > \hat{L}$. Now, consider

$$d_\infty(L) - d_0 = -(x_{10} - (\delta - r)x_{20}/\beta) + \alpha x_{20}\sqrt{\kappa}v(L)/\beta.$$

If $x_{10} - (\delta - r)x_{20}/\beta \leq 0$, the right-hand side of this equation is positive for all $L > \hat{L}$, from which the first part of the lemma follows. Let $x_{10} - (\delta - r)x_{20}/\beta > 0$. Then the right-hand side of the equation approaches ∞ as $L \downarrow \hat{L}$ and a negative number as $L \to \infty$ and it equals zero for a unique value of $L > \hat{L}$, say L^*. This proves the second part. □

Thus under the condition $x_{10} - (\delta - r)x_{20}/\beta \leq 0$ the steady state debt exceeds the initial debt for all values of L larger than the critical value $\hat{L}$. This condition can equivalently be written as $d_0 - \overline{d} \leq (\delta - r)x_{20}/\beta$. Recall that $(\delta - r)x_{20}/\beta$ is positive. So if the difference between initial and target debt is not too big, the moving horizon steady state debt is larger than the initial debt for each stable horizon length. Alternatively, if

$$\zeta := x_{10} - (\delta - r)x_{20}/\beta > 0,$$

or equivalently, $d_0 - \overline{d} > (\delta - r)x_{20}/\beta$, there exists another critical value of L, i.e. L^*, such that the moving horizon steady state debt is closer to its target than in the initial situation if and only if the moving horizon length is larger than L^*. Apparently, the difference between the initial and target debt should be sufficiently large for such a critical value to exist. Note that the number L^* is the unique root of the equation $d_\infty(L) = d_0$ in the interval $(\hat{L}, \infty)$. It is not possible to determine L^* analytically; a numerical computation is of course straightforward. In the sequel we assume that $\zeta > 0$. The next result states that the moving horizon steady state debt is decreasing in L. However, the target debt will never be reached. It also states that for different values of the horizon length, the corresponding moving horizon debt trajectories do not intersect.

Lemma 3.4.3 *The trajectory $t \mapsto d^{MH}(t; L)$ is constant for $L = L^*$ and decreasing for $L > L^*$. For any horizon lengths L_1 and L_2 with $L^* < L_1 < L_2$ we have*

$$\bar{d} + x_{20}(\delta - r)/\beta = \lim_{L\to\infty} d_\infty(L) < d_\infty(L_2) < d_\infty(L_1) < d_0. \tag{3.60}$$

Finally, the graphs of $t \mapsto d^{MH}(t; L_1)$ and $t \mapsto d^{MH}(t; L_2)$ do not intersect.

Proof Taking the derivative of d^{MH} with respect to t and using the facts $d_\infty(L^*) = d_0$ and $d_\infty(L) < d_0$ for all $L > L^*$ directly yields the first part of the proposition. The second part, i.e. (3.60), follows from the assumption $\delta - r > 0$, and the basic observations concerning the function v made at the beginning of the proof of the previous lemma. For the third part, assume there exists a $t_0 > 0$ with $d^{MH}(t_0; L_1) = d^{MH}(t_0; L_2)$. Then, according to the mean value theorem there must exist a number L_3 with $L_1 < L_3 < L_2$ so that $(\partial d^{MH}/\partial L)(t_0; L_3) = 0$, or, equivalently,

$$d'_\infty(L_3)\left(1 - e^{-h(L_3)t_0}\right) - (d_0 - d_\infty(L_3))h'(L_3)t_0e^{-h(L_3)t_0} = 0.$$

Observe that the left-hand side of this equation is negative, which is obviously a contradiction. □

In Figure 3.5 four graphs of the moving horizon government debt are drawn. Note that the limit $\lim_{L\to\infty} d_\infty(L)$ also equals $d_0 - \zeta$ as depicted in this figure. Thus $[d_0 - \zeta, d_0]$ indicates the range of the moving horizon steady state debt. Furthermore, the difference between the target and steady state debt exceeds $(\delta - r)x_{20}/\beta$ and for increasing values of L this difference decreases.

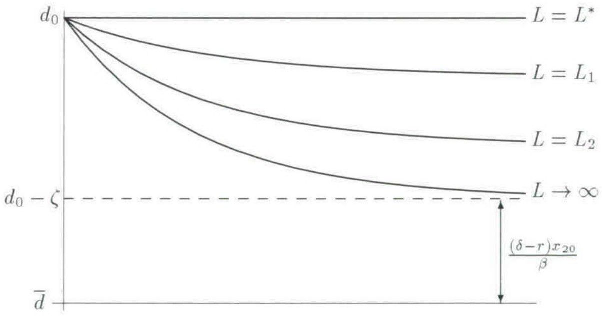


Figure 3.5: Moving Horizon Government Debt Trajectories ($L_1 < L_2$).

We summarize the results obtained so far in the following theorem.

Theorem 3.4.4 *Consider the government debt stabilization game*

$$\dot{d} = rd + f - m, \quad d(0) = d_0;$$
$$J_f = \int_0^\infty e^{-\delta t}\left((f-\overline{f})^2 + \kappa_1(d-\overline{d})^2\right)dt, \quad J_m = \int_0^\infty e^{-\delta t}\left((m-\overline{m})^2 + \kappa_2(d-\overline{d})^2\right)dt.$$

The parameters $r, d_0, \delta, \overline{f}, \overline{d}, \overline{m}, \kappa_1, \kappa_2$ are positive constants. Assume that $d_0 - \overline{d} > 0$, $r\overline{d} + \overline{f} - \overline{m} > 0$, and $\delta - r > 0$. There exists a unique moving horizon solution for each horizon length L. Let

$$\kappa := \kappa_1 + \kappa_2;$$
$$\alpha := \sqrt{(r-\delta/2)^2 + \kappa};$$
$$\hat{L} := \frac{1}{2\alpha}\log\frac{(\alpha + r - \delta/2)(\alpha+\delta/2)}{(\alpha - (r-\delta/2))(\alpha - \delta/2)}.$$

Then $\hat{L} > 0$ and the moving horizon closed-loop government debt trajectory, say $d^{MH}(t;L)$, reaches a steady state value if and only if $L > \hat{L}$. We have

$$d^{MH}(t;L) = d_\infty(L) + (d_0 - d_\infty(L))e^{-h(L)t}, \quad L > \hat{L}$$

with

$$h(L) := \alpha\tanh(\alpha L - \theta) - \delta/2;$$
$$d_\infty(L) := \overline{d} + \frac{x_{20}}{\beta}(\delta - r + \alpha\sqrt{\kappa}v(L));$$
$$v(L) := e^{-\delta L/2}/\left(\alpha\sinh(\alpha L - \theta) - (\delta/2)\cosh(\alpha L - \theta)\right);$$
$$\theta := \frac{1}{2}\log\frac{\alpha + r - \delta/2}{\alpha - r + \delta/2};$$
$$x_{20} := r\overline{d} + \overline{f} - \overline{m};$$
$$\beta := \kappa - r(\delta - r).$$

The value $d_\infty(L)$ is equal to the steady value of the moving horizon closed-loop government debt trajectory. Let

$$\zeta := d_0 - \overline{d} - (\delta - r)x_{20}/\beta.$$

Then, if $\zeta \leq 0$, the steady state government debt exceeds the initial debt (ands hence the target debt) for each $L > \hat{L}$. Assume further that $\zeta > 0$. Then there exists a unique root of the equation $d_\infty(L) = d_0$ in the interval $(\hat{L}, \infty)$. Denote this root by L^*. The steady state government debt *is smaller than the initial debt if and only if $L > L^*$. The larger the value of L is, the closer the steady state government debt is to its target value, but it is always strictly larger than the target value.*

Next, we determine the moving horizon strategy of the fiscal player. For that purpose we first determine the closed-loop moving horizon strategy $u_1^{MH}(x; L)$ corresponding to the "two-state variable" game, which is related to the government debt stabilization game through (3.29)-(3.33). From (3.9) and Proposition 3.3.5 we obtain

$$u_1^{MH}(x; L) = -B_1^T G_1(L) H^{-1}(L) x = -q_1 B_1^T G(L) H^{-1}(L) x.$$

Here we have $q_1 = \kappa_1$, the matrix B_1 is given by (3.33), and GH^{-1} is obtainable from (3.15). Indeed, this equation implies

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} G(L) H^{-1}(L) = \frac{1}{\kappa} \begin{bmatrix} r - \delta/2 - a_1(L) & 1 - a_2(L) \\ 0 & 0 \end{bmatrix},$$

which uniquely specifies the first row of GH^{-1}. Since B_1 has a zero on the second row, this is the only row of GH^{-1} we need. We find

$$u_1^{MH}(x; L) = \frac{\kappa_1}{\kappa} \left((a_1(L) - r + \delta/2) x_1 + (a_2(L) - 1) x_2 \right). \tag{3.61}$$

Instead of determining the moving horizon strategy of the fiscal player, we shall determine his corresponding fiscal control function, which we denote by $f^{MH}(t; L)$. An expression for this function can be found by replacing the state variables x_1 and x_2 in (3.61) by the moving horizon state trajectories which we have determined earlier. Indeed, they are uniquely specified by the initial value problem (3.51), (3.52). The solution of this problem is given by (3.54) and (3.53), which is expressed in terms of a_1, a_2, and a_4. For these closed-loop matrix functions we have determined the expressions (3.46), (3.50), and (3.48), respectively. Substituting them into the moving horizon state trajectories and into (3.61), and using (3.31) shows that the moving horizon closed-loop fiscal control trajectory can be written as (we shall only consider the stable case $L > \hat{L}$)

$$f^{MH}(t; L) = f_\infty(L) + (f_0(L) - f_\infty(L)) e^{-h(L)t}, \quad L > \hat{L}, \tag{3.62}$$

with

$$f_\infty(L) = \overline{f} - \frac{\kappa_1 x_{20}}{\beta} \left(1 + \frac{\alpha r}{\sqrt{\kappa}} v(L) \right), \tag{3.63}$$

$$f_0(L) = \overline{f} - \frac{\kappa_1}{\kappa} \left(r(d_0 - \overline{d}) + x_{20} + \zeta h(L) - \frac{\alpha x_{20} \sqrt{\kappa} e^{-\delta L/2}}{\beta \cosh(\alpha L - \theta)} \right). \tag{3.64}$$

Since $h(L) > 0$ we have $f^{MH}(0; L) = f_0(L)$ and $f^{MH}(t; L) \to f_\infty(L)$ as $t \to \infty$ for all $L > \hat{L}$; thus $f_0(L)$ and $f_\infty(L)$ denote the initial and steady state value of the moving horizon fiscal control trajectory, respectively. Recall that we have assumed that $\beta > 0$ and $\zeta > 0$. The first assumption was made in order to ensure that for sufficiently large values of the moving horizon length ($L > \hat{L}$), the government debt reaches a steady state value as time goes to infinity. The second assumption has been made in order to ensure that for sufficiently large stable values of the moving horizon length ($L > L^* > \hat{L}$), the moving horizon steady state government debt is closer to its target value than in the initial state. Let us further assume that $L \geq L^*$. In the next theorem we show that the moving horizon fiscal control trajectory is constant for $L = L^*$, i.e. in that case it equals the value

$$f^* := \overline{f} - \frac{\kappa_1}{\kappa}(rx_{10} + x_{20}). \tag{3.65}$$

In this theorem, it will furthermore be shown that this trajectory is increasing for $L > L^*$; as time evolves the value of the fiscal policy moves closer to its target value. The initial and steady state value are shown to be decreasing and increasing in L, respectively. Finally, we show that for two different values of the horizon length, the corresponding moving horizon fiscal control trajectories intersect exactly once. All the properties listed in the following theorem are illustrated by Figure 3.6.

Theorem 3.4.5 *Consider the government debt stabilization game from Theorem 3.4.4. Let all the assumptions made in this theorem be fulfilled. Furthermore, the notations introduced in Theorem 3.4.4 are also used here. The moving horizon fiscal control trajectory $f^{MH}(t; L)$ corresponding to the moving horizon strategy of the fiscal player is given by* (3.62)-(3.64). *Let the value f^* be defined by* (3.65) *and assume that $\zeta > 0$. Then, the trajectory $f^{MH}(t; L)$ is constant for $L = L^*$ (the constant value is f^*) and strictly increasing for $L > L^*$. Moreover, for any horizon lengths L_1 and L_2 with $L^* < L_1 < L_2$ we have*

$$f^* - \frac{\kappa_1}{\kappa}\zeta(\alpha - \delta/2) = \lim_{L\to\infty} f_0(L) < f_0(L_2) < f_0(L_1) < f^* \tag{3.66}$$

and

$$f^* < f_\infty(L_1) < f_\infty(L_2) < \lim_{L\to\infty} f_\infty(L) = f^* + \frac{\kappa_1}{\kappa}\zeta r. \tag{3.67}$$

Finally, there exists a unique $t > 0$ satisfying $f^{MH}(t; L_1) = f^{MH}(t; L_2)$.

Proof Since $d_\infty(L^*) = d_0$ we can derive an explicit expression for $v(L^*)$ from (3.57). Inserting this expression into (3.63) and (3.64), and using the identity

$$e^{-\delta L/2}/\cosh(\alpha L - \theta) = v(L)h(L)$$

yields $f_0(L^*) = f_\infty(L^*) = f^*$. This implies that $f^{MH}(t; L^*) = f^*$ for all $t \geq 0$. Since v is strictly decreasing it follows that f_∞ is strictly increasing. Next, we analyze the monotonicity of f_0. The derivative of f_0 is given by

$$f_0'(L) = -\frac{\kappa_1}{\kappa}h'(L)\left(\zeta - \frac{\alpha x_{20}\sqrt{\kappa}}{\beta}v(L)\right) + \frac{\kappa_1 \alpha x_{20}}{\sqrt{\kappa}\beta}h(L)v'(L).$$

In Lemma 3.4.2 we showed $d_\infty(L) < d_0$ for $L > L^*$. This inequality is equivalent to $\zeta - x_{20}\alpha\sqrt{\kappa}v(L)/\beta > 0$, implying that f_0 is strictly decreasing. The fact that $f_0(L^*) = f_\infty(L^*)$ combined with the monotonicity properties of f_0 and f_∞ implies that $f_0(L) < f_\infty(L)$ for all $L > L^*$. From this we conclude that $(\partial f^{MH}/\partial t)(t; L) = -(f_0(L) - f_\infty(L))h(L)e^{-h(L)t} > 0$, which completes the first part of the theorem. The second part of the theorem consists of (3.66) and (3.67). The limits for $L \to \infty$ directly follow from $v(L) \to 0$, $h(L) \to \alpha - \delta/2$ as $L \to \infty$. The other assertions in (3.66) and (3.67) follow from the proof of the first part of this theorem. For the third part, let $F(t) = f^{MH}(t; L_1) - f^{MH}(t; L_2)$. Then, obviously $F(0) > 0$ and $\lim_{t\to\infty} F(t) < 0$, showing the existence of a number t with the desired property. For the uniqueness, consider the equation $\dot{F}(t) = 0$, or equivalently,

$$e^{(h(L_2)-h(L_1))t} = \frac{h(L_2)(f_\infty(L_2) - f_0(L_2))}{h(L_1)(f_\infty(L_1) - f_0(L_1))}.$$

Since $h(L_2)(f_\infty(L_2) - f_0(L_2)) > h(L_1)(f_\infty(L_1) - f_0(L_1)) > 0$ and $h(L_2) > h(L_1)$ we observe that this equation has exactly one solution. This shows the uniqueness property. □

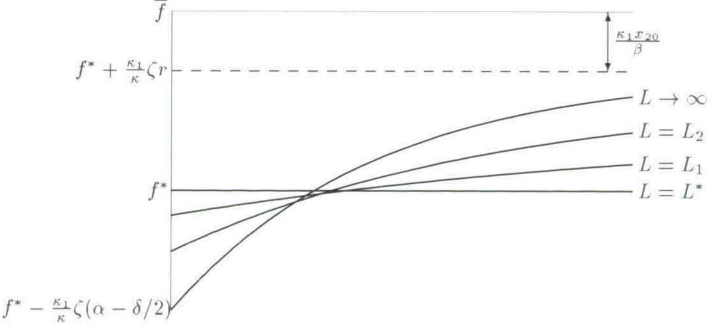


Figure 3.6: Moving Horizon Fiscal Control Trajectories ($L_1 < L_2$).

Remark 3.4.6 The limit of f_∞ for $L \to \infty$ can also be written as

$$\lim_{L\to\infty} f_\infty(L) = \overline{f} - \frac{\kappa_1 x_{20}}{\beta}. \tag{3.68}$$

The limits in (3.66) and (3.67) are written such as to show the maximal difference between initial and steady state moving horizon fiscal control values, respectively. The format in (3.68) shows furthermore that the target value $\overline{f}$ is never reached and that the difference between the steady state value of the moving horizon fiscal control trajectory and its target value always exceeds the value $\kappa_1 x_{20}/\beta$. □

Remark 3.4.7 Theorem 3.4.5 implies that for any $L > L^*$, a unique number $t > 0$ exists, which satisfies the equation $f^{MH}(t; L) = f^*$. This enables us to define a function ψ on (L^*, ∞) implicitly by

$$t = \psi(L) :\Leftrightarrow f^{MH}(t; L) = f^*. \tag{3.69}$$

Geometrically, the point $(\psi(L), f^*)$ is the intersection point of the curves corresponding to the equations $f = f(t, L)$ and $f = f^*$ in the (t, f)–plane. The equation $f^{MH}(t; L) = f^*$ can be written as

$$(r + h(L))\left(\zeta - \frac{x_{20}\alpha\sqrt{\kappa}}{\beta}v(L)\right)e^{-h(L)t} = r\left(\zeta - \frac{x_{20}\alpha\sqrt{\kappa}}{\beta}v(L)\right). \tag{3.70}$$

Now, $L > L^*$ implies that $\zeta - x_{20}\alpha\sqrt{\kappa}v(L)/\beta > 0$ (see also the proof of Theorem 3.4.5), which shows that this factor is unequal to zero. Consequently, ψ is given by

$$\psi(L) = \frac{1}{h(L)}\log\left(1 + \frac{h(L)}{r}\right), \quad L > L^*. \tag{3.71}$$

It can easily be seen that ψ is bounded and decreasing. The upper bound is reached for $L \downarrow L^*$ and can easily be determined numerically in concrete situations; the lower bound (reached for $L \to \infty$) is given by $(\alpha - \delta/2)^{-1}\log(1 + (\alpha - \delta/2)/r)$. □

Using the results of Theorem 3.4.5 and the discussion on ψ we visualize the behavior of the moving horizon fiscal control trajectory in Figure 3.6. Recall that the difference between the steady state value of the fiscal control and its target value always exceeds $\kappa_1 x_{20}/\beta$ for all $L \geq L^*$ and for increasing L this difference decreases. However, the difference between the initial value of the fiscal control and its target value is increasing for increasing L. Thus we see that the fiscal player benefits in the short run in the moving horizon solution in the sense that the difference between his primary fiscal deficits and the target value of this deficits is smaller than for instance in the open-loop Nash equilibrium. The smaller the moving horizon length, the smaller this difference becomes. In the long run the situation is exactly the other way around; the shorter the horizon length, the larger the difference between the steady state value of the fiscal deficits and the target value becomes[2]. But clearly, the moving horizon solution concept has not been introduced for the players to benefit from it in the long run.

Next, we determine the moving horizon monetary control trajectory, which will be denoted by $m^{MH}(t;L)$. Since $B_2 = -B_1$, the moving horizon strategies of the fiscal and monetary player differ by a multiplicative constant, equal to $-\kappa_2/\kappa_1$. Thus the moving horizon monetary control trajectory can be obtained from the moving horizon fiscal control trajectory. Indeed, we have

$$m^{MH}(t;L) = \overline{m} - \frac{\kappa_2}{\kappa_1}\left(f^{MH}(t;L) - \overline{f}\right), \quad L > \hat{L}. \tag{3.72}$$

Let $m_0(L) := m^{MH}(0;L)$, $m_\infty(L) := \lim_{t\to\infty} m^{MH}(t;L)$ and

$$m^* := \overline{m} + \frac{\kappa_2}{\kappa}(rx_{10} + x_{20}), \tag{3.73}$$

then the following theorem can be derived directly from Theorem 3.4.5.

Theorem 3.4.8 *Consider the government debt stabilization game from Theorem 3.4.4. Let all the assumptions made in this theorem be fullfilled. Furthermore, the notations introduced in the Theorems 3.4.4 and 3.4.5 are also used here. The moving horizon monetary control trajectory $m^{MH}(t;L)$ which corresponds to the moving horizon strategy of the monetary player is given by* (3.72). *Let the value m^* be defined by* (3.73) *and assume that $\zeta > 0$. Then, the trajectory $m^{MH}(t;L)$ is constant for $L = L^*$ (the constant value is m^*) and strictly decreasing for $L > L^*$. For any horizon lengths L_1 and L_2 with $L^* < L_1 < L_2$ we have*

$$m^* < m_0(L_1) < m_0(L_2) < \lim_{L\to\infty} m_0(L) = m^* + \frac{\kappa_2}{\kappa}\zeta(\alpha - \delta/2) \tag{3.74}$$

and

$$m^* - \frac{\kappa_2}{\kappa}\zeta r = \lim_{L\to\infty} m_\infty(L) < m_\infty(L_2) < m_\infty(L_1) < m^*. \tag{3.75}$$

Finally, there is a unique number $t > 0$ satisfying $m^{MH}(t;L_1) = m^{MH}(t;L_2)$.

Remark 3.4.9 The limit of m_∞ for $L \to \infty$ can also be written as $\overline{m} + \kappa_2 x_{20}/\beta$. This shows that the difference between the steady state value of the monetary moving horizon control trajectory and its target value always exceeds $\kappa_2 x_{20}/\beta$. □

[2]This can also be interpreted as follows: for a large value of the moving horizon length, the players put more effort on their initial control actions leading to better stability in the long run.

We visualize the behavior of the moving horizon monetary control trajectories in Figure 3.7 in a similar manner as we have done for the fiscal player in figure 3.6. Clearly also the monetary player benefits from the moving horizon solution concept in the short run in the same sense as the fiscal player does; see the discussion following Remark 3.4.7.

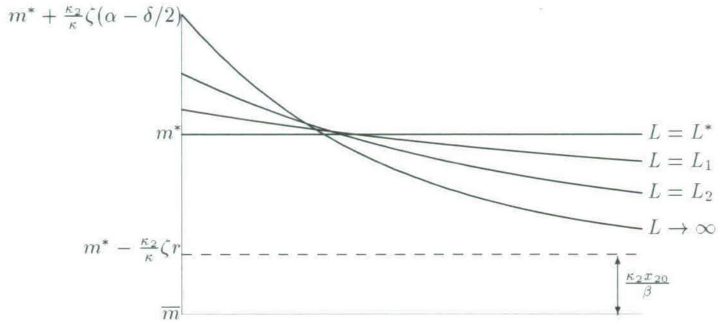


Figure 3.7: Moving Horizon Monetary Control Trajectories ($L_1 < L_2$).

3.5 Concluding Remarks

A new solution concept based on moving horizon control has been introduced in the area of time-invariant differential games. In this concept, the players determine at any point in time an open-loop Nash equilibrium for a finite planning horizon, and only play the initial equilibrium strategies. This is repeated continuously in time and in this way strategies defined on an infinite horizon with feedback information results. It is important to realize that the moving horizon solution concept is not an equilibrium concept. It is a method to control the system, which may be more in line with the paradigm of bounded rationality than the existing equilibrium concepts on finite or infinite horizons.

The moving horizon solution concept has been further analyzed in linear quadratic games. It was shown that if the moving horizon solution exists, it is unique. The open-loop control functions corresponding to the moving horizon solution converge in general to an open-loop Nash equilibrium of the infinite horizon game as the moving horizon length approaches infinity. Special attention has been paid to the scalar case. In this case the moving horizon solution could be determined analytically. It was shown that the rate of convergence of the optimal state increases for increasing values of the moving horizon length L. Moreover, for small values of L the closed-loop system can become unstable. The moving horizon length can be considered as an exogenous or as an endogenous parameter. In the latter case, a length L can be determined in such a way that the distance between the moving horizon solution and the closest feedback Nash equilibrium for the infinite horizon game is minimized. Additionally, this minimization process induces a selection criterion for a feedback Nash equilibrium in case of multiple equilibria.

The open-loop information structure of the finite-horizon game was chosen in order to make the resulting concept analytically tractable. From a conceptual point of view one could of course also choose the feedback Nash equilibrium as a solution concept for the local game. A first development in this direction can be found in [73]; in particular the issue of closed-loop stability is addressed in this reference.

Further concluding remarks can be found in Section 8.1.1. Preliminary versions of this chapter have been presented at the *Eighth International Symposium on Dynamic Games and Applications* (Maastricht, The Netherlands, 1998) and at the *Fifth International Conference of the Society for Computational Economics* (Boston, USA, 1999). The final version will appear in the *Journal of Economic Dynamics and Control* [24].

Chapter 4

Noncooperative Disturbance Decoupling

4.1 Introduction

In Chapter 2 we have introduced the notion of an input-output differential game. In such a game the players are influencing a dynamical system by choosing a certain input function and each player has his own output function. In general, this output function does not represent a cost function; it is a vector-valued function which depends on the state vector and the input functions of all the players. Given an initial state, the output function is a time trajectory in a certain function space. One could of course associate costs to this output trajectory by taking its norm; see for instance Chapter 6 of the present thesis.

In the one-player case control design objectives are sometimes directly linked to the output rather than to a cost function. In a model with a disturbance term the design objective could for instance be to decouple the output from the disturbance, and additionally to internally stabilize the closed-loop system. Inputs establishing such goals are usually not unique. It seems that in dynamic game theory decoupling problems have not received much attention so far. In this chapter, we consider a dynamic game version of the well-known disturbance decoupling problem [15, 118, 125] for time-invariant linear systems. With the practical significance of feedback Nash equilibria in mind, we assume that the information structure of the players is a linear time-invariant feedback pattern. For simplicity reasons we consider the situation where the output of each player is not directly influenced by the input of the other players, but only indirectly through the state variable.

We consider several solution concepts depending on whether the players do or do not cooperate and depending on the information that the players have about each other's strategies. The cooperative case immediately reduces to a one-player problem and can therefore easily be solved using standard results from geometric control theory. In the noncooperative case we distinguish between uniform and universal solvability of what we shall call *the noncooperative disturbance decoupling problem* (NDDP); a formal definition will be given in Section 4.3. In the uniform case the decoupling player is not aware of the other players' strategies. In Section 4.4 we derive necessary and sufficient conditions on the system parameters under which this player can find a feedback strategy achieving his decoupling purpose whatever the strategies of the other players are; under these condition we say that the NDDP has a *uniform* solution (the decoupling strategy is *independent* of the strategies of the other players). In the universal case the strategies of the other players are accessible to the decoupling player and he can use this information for his decoupling purposes. In Section 4.5 we derive necessary and sufficient conditions under which the decoupling player can always choose a decoupling strategy which depends on the strategies of the other players, i.e. under these conditions a *pointwise* solution of the NDDP exists *universally*.

4.2 Preliminaries

The NDDP will be formulated in the context of N-player deterministically disturbed linear time-invariant input-output differential games with the information structure of each player a linear time-invariant feedback pattern; see Section 2.3 for the terminology and notation. We assume that the matrices D_{ij} are zero. The equations of such a differential game are

$$\begin{aligned} \dot{x} &= \left(A + \sum_{i=1}^{N} B_i F_i\right) x + Ew, \quad x(0) = x_0; \\ z_i &= C_i x. \end{aligned}$$

It is a standing assumption that $E \neq 0$. The output of player i can be written as

$$z_i(t) = C_i e^{t\left(A+\sum_{i=1}^{N} B_i F_i\right)} x_0 + \int_0^t T_i(t-\tau) w(\tau) d\tau,$$

where $T_i(t) := C_i e^{t\left(A+\sum_{i=1}^{N} B_i F_i\right)} E$ is the closed-loop impulse response from the disturbance to the output of player i. The system is said to be *disturbance decoupled* for player i if $T_i(t) = 0$ for all $t \geq 0$.

Next, we briefly review some concepts from geometric control theory [15, 118] which are used in this chapter. Let $\mathcal{B}$ and $\mathcal{C}$ be subspaces of $\mathbb{R}^n$. A subspace $\mathcal{W}$ is said to be an $(A, \mathcal{B})$-controlled invariant if $A\mathcal{W} \subset \mathcal{W} + \mathcal{B}$. The maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$ is denoted by

$\max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$. It is well-known that in the one-player case the system is disturbance decoupled (for player 1) if and only if

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1).$$

In Section 4.5 we use the concept of *robust controlled invariance*; see [14] or [15, Section 6.5]. This geometric concept has originally been introduced in order to deal with robustness with respect to sudden changes in the parameters; sometimes referred to as hyper-robustness. A robust controlled invariant is defined as follows. Let A and $\mathcal{B}$ depend on a parameter $q \in Q \subset \mathbb{R}^r$. A subspace $\mathcal{W}$ is called a robust $(A(q), \mathcal{B}(q))$-controlled invariant relative to Q if $A(q)\mathcal{W} \subset \mathcal{W} + \mathcal{B}(q)$ for all $q \in Q$. The maximal robust $(A(q), \mathcal{B}(q))$-controlled invariant contained in $\mathcal{C}$ is denoted by $\max \mathcal{V}_R(A(q), \mathcal{B}(q), \mathcal{C})$. For the computation of this space the following algorithm is available. In this algorithm we denote by $f^{\leftarrow}(\mathcal{Y}_0)$ the inverse image of a set $\mathcal{Y}_0 \subset \mathcal{Y}$ under f, where f is a map with image contained in $\mathcal{Y}$.

Algorithm 4.2.1 [36] The subspace $\max \mathcal{V}_R(A(q), \mathcal{B}(q), \mathcal{C})$ coincides with the last term of the sequence

$$\mathcal{Z}_0 := \mathcal{C},$$
$$\mathcal{Z}_i := \mathcal{C} \cap \left(\bigcap_{q \in Q} A(q)^{\leftarrow}(\mathcal{Z}_{i-1} + \mathcal{B}(q)) \right), \quad i = 1, \ldots, k,$$

where the value of k is determined by the condition $\mathcal{Z}_k = \mathcal{Z}_{k-1}$. It can be shown that $k \leq n$. $\square$

The concept of a robust controlled invariant can be used to solve the *hyper-robust disturbance localization problem* [15, Section 6.5.1]. Let A and B_1 depend on the parameter q. Then, in a one-player context the hyper-robust localization problem is known as the problem of finding for each $q \in Q$ a feedback map $F_1(q)$ such that the system is disturbance decoupled (for player 1). This problem is said to be solvable if such a feedback map exists for each $q \in Q$. Necessary and sufficient conditions are presented in the following theorem.

Theorem 4.2.2 [15, Theorem 6.5-1] *The hyper-robust disturbance localization problem is solvable if and only if*

$$\operatorname{im} E \subset \max \mathcal{V}_R(A(q), \operatorname{im} B_1(q), \ker C_1).$$

4.3 Problem Formulations

In game theory one usually distinguishes between cooperative and noncooperative solutions. The cooperative version of the N-player disturbance decoupling problem is to find feedback maps F_i such that the system is disturbance decoupled for all the players. It follows immediately that this cooperative problem is solvable if and only if

$$\operatorname{im} E \subset \max \mathcal{V}\left(A, \sum_{i=1}^{N} \operatorname{im} B_i, \bigcap_{i=1}^{N} \ker C_i\right).$$

In the rest of this chapter we consider with noncooperative versions of the N-player disturbance decoupling problem.

In the noncooperative setting, different problems can be posed depending on whether or not players choose their feedback policies in a certain order and are aware of each other's decisions. We shall fix one player and consider the situations in which this player is or is not aware of the other players' chosen feedback policies. This decoupling player faces the problem of finding a feedback control law such that his output is decoupled from the disturbance even in the presence of other noncooperative players. We shall refer to this problem as the *noncooperative disturbance decoupling problem* (NDDP). To be precise, we consider the following two solution concepts. We say that the NDDP is *uniformly* solvable for player i if there exists a feedback matrix F_i such that the system is disturbance decoupled for player i for all F_j, $j \neq i$. This notion of solvability will be studied in Section 4.4. In contrast to this, we say that the NDDP is *universally* solvable for player i if for each $(F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_N)$ there exists an F_i such that the system is disturbance decoupled for player i. This notion of solvability will be studied in Section 4.5. The distinction we make here is in line with studying different types of information structure, as is usually done in dynamic game theory.

Without loss of generality we take $N = 2$ and consider solvability of the NDDP for player 1.

4.4 Uniform Solvability of the NDDP

The decoupling problem of this section can be formally defined as follows.

Problem 4.4.1 Find a feedback map F_1 such that the system

$$\dot{x} = (A + B_1 F_1 + B_2 F_2)x + Ew, \quad z_1 = C_1 x \tag{4.1}$$

is disturbance decoupled for player 1 for all feedback maps F_2. □

As pointed out in the previous section we say that the NDDP is *uniformly solvable* for player 1 if Problem 4.4.1 has a solution. The aim here is to find necessary and sufficient conditions on the system parameters for this type of solvability.

Assume that F_1 solves Problem 4.4.1. We then have that $T_1(t) = 0$ for all F_2, or, equivalently

$$C_1(A + B_1F_1 + B_2F_2)^k E = 0, \text{ for all } F_2 \text{ and for all } k \geq 0. \tag{4.2}$$

By taking $F_2 = 0$, it follows that

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1). \tag{4.3}$$

Obviously, this is a necessary condition for solvability of Problem 4.4.1. In Lemma 4.4.3 we present a second necessary condition. For this we need the following result.

Lemma 4.4.2 *Let X_1 and X_2 be $n \times n$ matrices, then for each $k \in I\!N$,*

$$(X_1 + X_2)^k = X_1^k + \sum_{j=0}^{k-1} X_1^j X_2 (X_1 + X_2)^{k-1-j}. \tag{4.4}$$

Proof We prove this lemma by induction. For $k = 1$ the claim is obvious. Assume that (4.4) holds for a certain number k, then

$$\begin{aligned}(X_1 + X_2)^{k+1} &= \left(X_1^k + \sum_{j=0}^{k-1} X_1^j X_2 (X_1 + X_2)^{k-1-j} \right)(X_1 + X_2) = \\ &= X_1^{k+1} + X_1^k X_2 + \sum_{j=0}^{k-1} X_1^j X_2 (X_1 + X_2)^{k-j} = \\ &= X_1^{k+1} + \sum_{j=0}^{k} X_1^j X_2 (X_1 + X_2)^{k-j}.\end{aligned}$$

□

Lemma 4.4.3 *If there exists an F_1 for which (4.2) holds, then*

$$C_1(A + B_1F_1)^l B_2 = 0, \textit{ for all } l \geq 0. \tag{4.5}$$

Proof We prove this lemma by induction. Setting $k = 1$ in (4.2), we find $C_1(A + B_1F_1 + B_2F_2)E = 0$ for all F_2. Taking $F_2 = 0$ yields $C_1(A+B_1F_1)E = 0$ so that we have $C_1B_2F_2E = 0$ for all F_2. Since $E \neq 0$ this gives us $C_1B_2 = 0$, i.e. (4.5) holds for $l = 0$. Now, fix an $l_0 \geq 0$ and assume that (4.5) holds for all $l = 0, \ldots, l_0$. Take $k = l_0 + 2$ and $F_2 = 0$ in (4.2) to get $C_1(A+B_1F_1)^{l_0+2}E = 0$. Use this and the formula (which follows from (4.4) with $X_1 = A+B_1F_1$ and $X_2 = B_2F_2$)

$$\begin{aligned}(A + B_1F_1 + B_2F_2)^{l_0+2} =&(A + B_1F_1)^{l_0+2} + \\ &+ \sum_{j=0}^{l_0+1}(A + B_1F_1)^j B_2F_2(A + B_1F_1 + B_2F_2)^{l_0+1-j}\end{aligned}$$

to see that (4.2) (with $k = l_0 + 2$) implies

$$\sum_{j=0}^{l_0+1} C_1(A + B_1F_1)^j B_2F_2(A + B_1F_1 + B_2F_2)^{l_0+1-j}E = 0, \text{ for all } F_2.$$

Using the induction assumption we see that this reduces to $C_1(A + B_1F_1)^{l_0+1}B_2F_2E = 0$ for all F_2. Since $E \neq 0$ it follows that $C_1(A + B_1F_1)^{l_0+1}B_2 = 0$, which completes the proof. □

So the feedback mapping F_1 that solves Problem 4.4.1 also satisfies (4.5). As a consequence the system matrices must satisfy

$$\operatorname{im} B_2 \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1).$$

In the following theorem it is stated that the two necessary conditions (4.3) and (4.5) are also sufficient for the uniform solvability of the NDDP for player 1.

Theorem 4.4.4 *The NDDP is uniformly solvable for player 1 if and only if*

$$\operatorname{im} \begin{bmatrix} B_2 & E \end{bmatrix} \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1). \tag{4.6}$$

Proof The necessity has already been proven. For the sufficiency, it suffices to consider the control input of player 2 as an extra disturbance. Indeed, assume that (4.6) holds and write $\mathcal{W} := \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1)$. Since $\mathcal{W}$ is an $(A, \operatorname{im} B_1)$-controlled invariant, there exists an F_1 such that $\mathcal{W}$ is an $A + B_1F_1$-invariant, so that

$$(A + B_1F_1 + B_2F_2)\mathcal{W} \subset \mathcal{W} + \operatorname{im} B_2 \subset \mathcal{W}.$$

Together with (4.6) this shows that the closed-loop system is disturbance decoupled for player 1. □

Formula (4.6) provides a compact necessary and sufficient condition for the existence of a linear state feedback control for the decoupling player that achieves disturbance decoupling for him whatever the other player will do. Under this condition, a decoupling feedback can be found by standard techniques from geometric control theory (see also the proof of the sufficiency part of Theorem 4.4.4).

4.5 Universal Solvability of the NDDP

The decoupling problem of this section can be formally defined as follows.

Problem 4.5.1 For a given feedback matrix F_2, find a feedback matrix F_1 such that the system (4.1) is disturbance decoupled for player 1. □

As pointed out in Section 4.3 we say that the NDDP is *universally solvable* for player 1 if Problem 4.5.1 has a solution for each F_2. The aim of the present section is to find necessary and sufficient conditions for this type of solvability. Individual solutions of Problem 4.5.1 (if they exist) can be found by standard techniques from geometric control theory.

If the NDDP is universally solvable for player 1, he is able to respond to any control of the other player in such a way that he can decouple his output from the disturbance. In particular, if the other player chooses $F_2 = 0$, player 1 can decouple if and only if

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1). \tag{4.7}$$

Obviously, this condition is a necessary condition for the universal solvability of the NDDP for player 1.

For a given feedback matrix F_2 the system equations are

$$\begin{aligned} \dot{x} &= (A + B_2F_2)x + B_1u_1 + Ew, \\ z_1 &= C_1x. \end{aligned}$$

Although F_2 is not manipulable for the decoupling player, it is accessible for his decoupling purposes. We recognize this problem as a particular case of the hyper-robust disturbance localization problem. According to Theorem 4.2.2 the problem for the decoupling player is solvable if and only if

$$\operatorname{im} E \subset \max \mathcal{V}_R(A + B_2\cdot, \operatorname{im} B_1, \ker C_1), \tag{4.8}$$

i.e. the maximal robust $(A + B_2\cdot, \operatorname{im} B_1)$-controlled invariant relative to $I\!R^{m_2\times n}$ contained in $\ker C_1$ (we write a dot instead of an F_2 in order to indicate that the robustness needs to be interpreted with respect to this parameter). According to Algorithm 4.2.1 this space coincides with the last term of the sequence

$$\begin{aligned}
\mathcal{Z}_0 &:= \ker C_1, \\
\mathcal{Z}_i &:= \ker C_1 \cap \left(\bigcap_{F_2} (A + B_2F_2)^{\leftarrow}(\mathcal{Z}_{i-1} + \operatorname{im} B_1) \right), \quad i = 1, \dots, k.
\end{aligned}$$

Now, assume that Problem 4.5.1 has a solution for each F_2. Then, since $E \neq 0$ and according to (4.8), there exists an $x_0 \neq 0$ in the space $(A + B_2F_2)^{\leftarrow}(\mathcal{Z}_i + \operatorname{im} B_1)$ for all F_2 and for all $i = 0, 1, \dots$. Equivalently, $(A+B_2F_2)x_0 \in \mathcal{Z}_i + \operatorname{im} B_1$ for all F_2 and for all $i = 0, 1, \dots$. Because this holds for all F_2 and because $x_0 \neq 0$, we have $Ax_0 + b_2 \in \mathcal{Z}_i + \operatorname{im} B_1$ for all $b_2 \in \operatorname{im} B_2$ and for all $i = 0, 1, 2, \dots$. But this implies that

$$\operatorname{im} B_2 \subset \mathcal{Z}_i + \operatorname{im} B_1, \quad i = 0, 1, \dots . \tag{4.9}$$

We have now shown that universal solvability of the NDDP for player 1 implies (4.7) and (4.9). The latter condition is however not yet in terms of the system matrices. We will remedy this after the following lemma.

Lemma 4.5.2 *Consider the sequence $\mathcal{V}_i$, defined by*

$$\begin{aligned}
\mathcal{V}_0 &:= \ker C_1, \\
\mathcal{V}_i &:= \ker C_1 \cap A^{\leftarrow}(\mathcal{V}_{i-1} + \operatorname{im} B_1), \quad i = 1, \dots, k.
\end{aligned}$$

Then we have the inclusion $\mathcal{Z}_i \subset \mathcal{V}_i$ for all i.

Remark 4.5.3 The last term of the sequence $\mathcal{V}_i$ equals the maximal $(A, \operatorname{im} B_1)$-controlled invariant contained in $\ker C_1$. □

Proof of Lemma 4.5.2 We prove the lemma by induction. Clearly, the statement holds for $i = 0$. Assume that $\mathcal{Z}_i \subset \mathcal{V}_i$ for a certain i. According to the definition of the subspaces $\mathcal{Z}_i$ we have $\mathcal{Z}_{i+1} \subset \ker C_1$ and $\mathcal{Z}_{i+1} \subset (A + B_2F_2)^{\leftarrow}(\mathcal{Z}_i + \operatorname{im} B_1)$ for all F_2, so in particular for $F_2 = 0$. This results in

$$A\mathcal{Z}_{i+1} \subset \mathcal{Z}_i + \operatorname{im} B_1 \subset \mathcal{V}_i + \operatorname{im} B_1$$

and hence, $\mathcal{Z}_{i+1} \subset A^{\leftarrow}(\mathcal{V}_i + \operatorname{im} B_1) \cap \ker C_1 = \mathcal{V}_{i+1}$, which completes the proof. $\square$

Applying this lemma to (4.9) shows that $\operatorname{im} B_2 \subset \mathcal{V}_i + \operatorname{im} B_1$ for all i and by taking the last term in this sequence (as noted in Remark 4.5.3) we arrive at a necessary condition in terms of the system matrices. In the following theorem we claim that this condition together with condition (4.7) is actually also sufficient for universal solvability of the NDDP for player 1.

Theorem 4.5.4 *The NDDP is universally solvable for player 1 if and only if*

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1), \tag{4.10}$$

$$\operatorname{im} B_2 \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1) + \operatorname{im} B_1. \tag{4.11}$$

Proof The necessity has already been proven. The sufficiency can be concluded by considering the input of player 2 as an accessible disturbance (cf. [125, Exercise 4.10], [15, page 225]). Specifically, let $F_2 \in \mathbb{R}^{m_2 \times n}$ and write $\mathcal{W} := \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1)$. According to (4.11), we have $\operatorname{im} E \subset \mathcal{W} \subset \ker C_1$. Furthermore, since $\mathcal{W}$ is an $(A, \operatorname{im} B_1)$-controlled invariant and because of (4.11), we also have $(A+B_2F_2)\mathcal{W} \subset \mathcal{W} + \operatorname{im} B_1$, so that $\mathcal{W}$ is also an $(A+B_2F_2, \operatorname{im} B_1)$-controlled invariant; hence there exists a mapping F_1 such that $\mathcal{W}$ is an $(A + B_2F_2 + B_1F_1)$-invariant. $\square$

Formula (4.10) and (4.11) provide compact necessary and sufficient conditions for the decoupling player to be able to respond to the action of the other player always in such a way that his output is decoupled from the disturbance.

4.6 Concluding Remarks

The aim of this chapter has been to define some disturbance decoupling problems in the context of differential games. Necessary and sufficient conditions for the solvability of these problems in terms of the system data have been derived; consequently, one can easily verify whether the considered problems are solvable for a given system.

The idea of considering games in which the players follow certain control objectives that are not necessarily given by cost functions can be carried further in many ways. In the context of decoupling, extensions can for instance be made in the direction of incorporating stability and to the case where the players are not able to completely observe the state and/or each other's actions. The notion of equilibrium could be incorporated to a greater extent by considering games in which the players know that the other players will look for a decoupling feedback but are not informed about each other's decisions. The two noncooperative problems considered in this chapter provide necessary and sufficient conditions respectively for a decoupling solution to exist in this situation.

Further concluding remarks can be found in Section 8.1.2. This chapter has been published in the journal *Systems & Control Letters* [27].

Chapter 5

Infinite-Horizon Feedback Nash Equilibria

5.1 Introduction

Due to its practical significance, the feedback Nash equilibrium concept is a popular solution concept in the field of dynamic games [10, 127]; application fields are for instance macro-economic policy coordination problems [3, 93, 115], resource games [35, 105], environmental economics [86, 100], and duopoly theory [49, 120]. Considering the work of Başar and Olsder [10] as the main basic reference in the field of dynamic game theory, [10, Definition 6.6] is the best definition for the feedback Nash equilibrium concept in the context of a finite-horizon differential game where the information structure of the players is a memoryless perfect state pattern or a closed-loop perfect state pattern[1]. In this definition a set of strategies is called a feedback Nash equilibrium solution if there exists a set of value functions satisfying the usual relations. This definition has the advantage that the information nonuniqueness problem [7] is eliminated and that equilibrium strategies are strongly time consistent [8]. Furthermore, feedback Nash equilibrium strategies only depend on the time variable and the state at time t; hence, the concept can also be used for games with a feedback information structure. In a linear-quadratic game with nonnegative state and control weighting matrices, one feedback Nash equilibrium is generated by a set of matrix valued functions which is a solution of a system of coupled matrix Riccati differential equations provided that this solution exists and that each matrix valued function is positive semi-definite [10, Corollary 6.5].

A definition of a feedback Nash equilibrium concept for infinite-horizon differential games is not presented by Başar and Olsder. Nevertheless, for a linear-quadratic game with nonnegative state

[1] This information pattern consists at time t of the complete state history from time 0 to time t.

and control weighting matrices, they show that any solution of a system of coupled algebraic Riccati equations that satisfies certain stabilizability and detectability conditions, generates a Nash equilibrium [10, Proposition 6.8]. In this equilibrium the strategies are linear time-invariant state feedback strategies and the resulting closed-loop matrix is stable. As noted by the authors, a disadvantage of this proposition is that the stabilizability and detectability conditions are indirect. Furthermore, the question whether all equilibria of this form, i.e. linear time-invariant state feedback strategies, are obtained from this system of coupled algebraic Riccati equations is not answered by this proposition. In the present chapter we remedy these disadvantages by showing that all equilibria in the class of linear time-invariant state feedback strategies yielding a stable closed-loop system, are obtained from the system of coupled algebraic Riccati equations. No detectability or stabilizability conditions are required for this statement. Furthermore, the state and cross control weighting matrices are not necessarily required to be positive semi-definite.

In the context of an N-player infinite-horizon linear-quadratic differential game we shall define a deterministic feedback Nash equilibrium as an N-tuple of linear time-invariant state feedback strategies stabilizing the closed-loop system such that if each player implements this strategy no player has the incentive to unilaterally deviate within this class of stabilizing feedback strategies (Definition 5.3.1 in Section 5.3). The corresponding equilibrium feedback matrices are required to be independent of the initial state which is in line with a feedback information pattern. The main result of this chapter is that all the deterministic feedback Nash equilibria are characterized by the solutions of the system of coupled algebraic Riccati equations which satisfy a stability condition (Theorem 5.3.2). This result is a straightforward generalization from a corresponding one-player equivalence statement, which can be formulated as follows. A linear, initial-state independent, time-invariant, internally stabilizing state feedback law which minimizes a quadratic cost functional exists if and only if it is generated by the stabilizing solution of the corresponding algebraic Riccati equation (Corollary 5.2.5 of Section 5.2). The minimization problem differs with respect to most of the linear-quadratic literature in that the state weighting matrix is not required to be positive semi-definite. For this reason the problem is sometimes referred to as the indefinite zero-endpoint linear-quadratic control problem. In a zero-endpoint problem it is explicitly required that the state converges to zero as time tends to infinity. This in contrast to a free-endpoint problem where this requirement is dropped [117]. The control weighting matrix in the quadratic criterion is assumed to be positive definite. For this reason the minimization problem is called regular. This in contrast to singular problems, where this matrix is not positive definite [33, 59, 124]. The regular indefinite zero-endpoint problem has been studied by several authors [92, 112, 119, 123]. In Section 5.2 we start with a short review of an important result from this literature (Theorem 5.2.2). Next, the equivalence statement is derived of which especially the necessity part is innovative in the linear-quadratic literature.

The optimal state feedback law which minimizes a quadratic infinite-horizon cost function subject to a deterministic linear constraint also minimizes the expected value of the average cost per time unit subject to a stochastic linear constraint [39, Theorem 5.4.4]. This result is based on the certainty equivalence principle, first stated in [111, 116] and generally formulated in [121]. The optimal state feedback law, which is obtained from the stabilizing solution of an algebraic Riccati equation, is in particular independent of the variance of the stochastic process. Using the equivalence statement of Section 5.2 we show in Section 5.4 that the converse statement is also true, i.e. if there exists a linear variance-independent state feedback law minimizing the expected value of the average cost per time unit, then it must be generated by the stabilizing solution of the same algebraic Riccati equation. This result can straightforwardly be extended to the N-player case, i.e. in Section 5.5 the concept of a stochastic variance-independent feedback Nash equilibrium is defined in the context of stochastic differential games. Based on the certainty equivalence principle we conclude in Theorem 5.5.2 that the set of deterministic feedback Nash equilibria coincides with the set of stochastic variance-independent feedback Nash equilibria.

In the stochastic LQ literature it is usually assumed that the state and control weighting matrices in the criterion are positive semi-definite and positive definite, respectively. The material in Sections 5.4 and 5.5 is presented without assuming that the state weighting matrix is positive semi-definite. However, the control weighting matrix is assumed to be positive definite. Stochastic problems with this matrix indefinite have obtained quite some attention lately [31, 80, 103, 104] due to a finding [31] that a class of stochastic LQ problems with an indefinite control matrix is well-posed. This in contrast to a deterministic LQ problem, which is ill-posed if this matrix is not positive semi-definite.

The chapter ends in Section 5.6 with an analysis of the two-player scalar case with zero cross control weights. It is geometrically shown that if both players have a negative state weight in their cost functional, there exist situations in which no feedback Nash equilibrium exists. If one of the players has a positive state weight, an equilibrium always exists. Geometric examples of situations with zero, one, two, or three equilibria are presented and it is indicated how the number of equilibria depends on the parameters.

5.2 An Equivalence Result in LQ Control Theory

In this section we study the following problem.

Problem 5.2.1 Consider the linear system

$$\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{5.1}$$

with (A, B) stabilizable, and a quadratic cost functional $J : \mathcal{F} \times I\!R^n \to I\!R$, defined by

$$J(F, x_0) = \int_0^\infty (x^T Q x + u^T R u) dt \tag{5.2}$$

where the state x follows from (5.1) with $u = Fx$. The state weighting matrix Q is symmetric and the control weighting matrix R is positive definite. Determine, if it exists, for each initial state x_0, a feedback matrix $\overline{F} \in \mathcal{F}$ such that $J(\overline{F}, x_0) \leq J(F, x_0)$ for all $F \in \mathcal{F}$. □

This problem is closely related to the regular indefinite zero-endpoint linear quadratic problem [119, 123], which can be formulated as follows. In addition to the infinite-horizon cost functional (5.2), consider for each $T > 0$ the finite-horizon cost functional $J_T : L^m_{2,loc}(0, \infty) \times I\!R^n \to I\!R$, defined by

$$J_T(u, x_0) = \int_0^T (x^T Q x + u^T R u) dt \tag{5.3}$$

where x is the solution of (5.1). The matrices Q and R are as in Problem 5.2.1. Define for each initial state the control class

$$\begin{aligned} \mathcal{U}(x_0) = \Big\{ u \in L^m_{2,loc}(0, \infty) \,\Big|\, & \lim_{T \to \infty} J_T(u, x_0) \text{ exists in } I\!R \cup \{-\infty, \infty\}, \\ & \lim_{t \to \infty} x(t) = 0, \text{ with } x \text{ the solution of (5.1)} \Big\}. \end{aligned} \tag{5.4}$$

The regular indefinite zero-endpoint linear quadratic control problem is the problem of minimizing $\lim_{T \to \infty} J_T(u, x_0)$ over the class of control functions $\mathcal{U}(x_0)$. The optimal value of the cost functional is denoted by $\overline{J}(x_0)$, i.e.

$$\overline{J}(x_0) := \inf_{u \in \mathcal{U}(x_0)} \lim_{T \to \infty} J_T(u, x_0). \tag{5.5}$$

It is not guaranteed that this infimum exists. The following theorem provides conditions for this to hold and moreover, it provides a solution for the regular indefinite zero-endpoint linear quadratic problem under these conditions. This theorem was proven by Willems [123] (see also [119, Theorem 8.8.2]).

Theorem 5.2.2 *Let (A, B) be controllable and define $S := BR^{-1}B^T$. Then $\overline{J}(x_0)$ is finite for all $x_0 \in \mathbb{R}^n$ if and only if there exists a real symmetric solution X of the algebraic Riccati equation*

$$Q + A^T X + XA - XSX = 0. \tag{5.6}$$

Assume that this condition holds. Then the smallest and largest real symmetric solution of (5.6) exist. Denote them by X^- and X^+, respectively. Then,

(i) $\overline{J}(x_0) = x_0^T X^+ x_0$ for all $x_0 \in \mathbb{R}^n$;

(ii) for all $x_0 \in \mathbb{R}^n$ there exists an optimal control function $\overline{u}$, i.e. a control function for which $\lim_{T\to\infty} J_T(\overline{u}, x_0) = \overline{J}(x_0)$, if and only if $X^+ - X^- > 0$;

(iii) if $X^+ - X^- > 0$, then for each x_0 there exists a unique optimal control function $\overline{u}$ and moreover, this control function can be written as the linear state feedback control law $\overline{u} = -R^{-1}B^T X^+ x$.

From this theorem we deduce that the feedback matrix $-R^{-1}B^T X^+$ is a solution for Problem 5.2.1 if (A, B) is controllable and if the algebraic Riccati equation (5.6) has a real symmetric solution and the difference between its largest and smallest solution is positive definite. Note that this feedback matrix is indeed an element of $\mathcal{F}$, since the control functions in the control class $\mathcal{U}(x_0)$ are by definition such that the corresponding state converges to zero as time tends to infinity. Thus X^+ is the stabilizing solution of the algebraic Riccati equation (5.6). Furthermore, note that the solution $-R^{-1}B^T X^+$ of Problem 5.2.1 is independent of the initial state x_0. In the following analysis it is shown that an initial state-independent solution of Problem 5.2.1 exists if and only if the algebraic Riccati equation has a stabilizing solution. This is done without assuming that (A, B) is controllable.

Let X be a real symmetric $n \times n$ matrix, then

$$\begin{aligned} J(F, x_0) &= \int_0^\infty \left(x^T(Q + F^T RF)x + \frac{d}{dt}x^T Xx - \frac{d}{dt}x^T Xx \right) dt = \\ &= x_0^T X x_0 + \int_0^\infty \left(x^T(Q + A^T X + XA)x + x^T F^T RFx + 2x^T F^T B^T Xx \right) dt, \end{aligned}$$

which holds because, due to the stability of $A + BF$, $x(t) \to 0$ for $t \to \infty$. Thus the completion of the square

$$x^T F^T RFx + 2x^T F^T B^T Xx = \left|(F + R^{-1}B^T X)x\right|_R^2 - x^T XSXx$$

with $S := BR^{-1}B^T$, yields

$$J(F, x_0) = x_0^T X x_0 + \int_0^\infty \left(x^T(Q + A^T X + XA - XSX)x + \left|(F + R^{-1}B^T X)x\right|_R^2 \right) dt.$$

Hence, if X satisfies the algebraic Riccati equation

$$Q + A^T X + XA - XSX = 0, \tag{5.7}$$

then

$$J(F, x_0) = x_0^T X x_0 + \int_0^\infty \left|(F + R^{-1}B^T X)x\right|_R^2 dt. \tag{5.8}$$

This formula shows that the feedback matrix $-R^{-1}B^T X$ minimizes $J(F, x_0)$. It is not guaranteed that this feedback matrix is an element of $\mathcal{F}$. However, if X is the stabilizing solution of the algebraic Riccati equation (5.7), this is ensured. We arrive at the following result (which follows in the controllable case from Willems [123]; see Theorem 5.2.2).

Theorem 5.2.3 *Consider Problem 5.2.1. Define $S := BR^{-1}B^T$. If the algebraic Riccati equation* (5.7) *has a stabilizing solution X, the feedback matrix $\overline{F} := -R^{-1}B^T X$ minimizes $J(F, x_0)$ for each initial state x_0.*

The converse statement of Theorem 5.2.3 is also true. This is formulated in the next theorem.

Theorem 5.2.4 *Consider Problem 5.2.1. Define $S := BR^{-1}B^T$. If there exists an $\overline{F} \in \mathcal{F}$ which minimizes $J(F, x_0)$ uniformly in x_0, i.e. $J(\overline{F}, x_0) \le J(F, x_0)$ for all $F \in \mathcal{F}$ and for all $x_0 \in \mathbb{R}^n$, the algebraic Riccati equation* (5.7) *has a stabilizing solution.*

Proof This theorem is shown by means of a differentiation argument applied to J. See Section 2.1 for details concerning matrix differentiation. Note that since eigenvalues depend continuously on the corresponding matrix elements [61, Appendix D], the set $\mathcal{F}$ is an open set. Define $\varphi : \mathcal{F} \to \mathbb{R}^{n \times n}$ by $\varphi : F \mapsto P$, where P is the unique solution of the Lyapunov equation

$$(A + BF)^T P + P(A + BF) = -Q - F^T RF.$$

Then $J(F, x_0) = x_0^T \varphi(F) x_0$. The smoothness of the coefficients in a Lyapunov equation is preserved by the solution of this equation [76, Section 5.4], which implies that J is differentiable with respect to F. By assumption, $J(F, x_0)$ attains its minimal value at $\overline{F}$ for each x_0. Thus [81, Section 7.4, Theorem 1] $\delta_1 J(\overline{F}, x_0; \Delta F) = 0$ for each increment ΔF and for each x_0. Since $\delta_1 J(\overline{F}, x_0; \Delta F) = x_0^T \delta\varphi(\overline{F}; \Delta F)x_0$, it follows that $\delta\varphi(\overline{F}; \Delta F) = 0$ for all increments ΔF. Hence

$$\partial\varphi(\overline{F}) = 0. \tag{5.9}$$

Define the transformation $\Phi : \mathcal{F} \times I\!R^{n\times n} \to I\!R^{n\times n}$ by

$$\Phi(F, P) = (A + BF)^T P + P(A + BF) + Q + F^T RF.$$

By definition, we have $\Phi(F, \varphi(F)) = 0$ for all $F \in \mathcal{F}$. Taking the derivative of this equality and applying the chain rule yields

$$\partial_1\Phi(F, \varphi(F)) + \partial_2\Phi(F, \varphi(F))\partial\varphi(F) = 0 \text{ for all } F \in \mathcal{F}.$$

By substituting $F = \overline{F}$ in this equality, and using (5.9), it follows that $\partial_1\Phi(\overline{F}, \varphi(\overline{F})) = 0$, or, equivalently,

$$\delta_1\Phi(\overline{F}, \varphi(\overline{F}); \Delta F) = 0 \text{ for all } \Delta F. \tag{5.10}$$

The differential of Φ with respect to its first argument with increment ΔF can easily be computed. Indeed,

$$\begin{aligned}
\delta_1\Phi(F, P; \Delta F) &= \lim_{\alpha\to 0} \frac{\Phi(F + \alpha\Delta F, P) - \Phi(F, P)}{\alpha} = \\
&= \lim_{\alpha\to 0} \frac{\alpha\Delta F^T(RF + B^T P) + \alpha(F^T R + PB)\Delta F + \alpha^2\Delta F^T R\Delta F}{\alpha} = \\
&= \Delta F^T(B^T P + RF) + (PB + F^T R)\Delta F.
\end{aligned}$$

Consequently, using (5.10),

$$\Delta F^T(B^T\varphi(\overline{F}) + R\overline{F}) + (\varphi(\overline{F})B + \overline{F}^T R)\Delta F = 0 \text{ for all } \Delta F.$$

Thus, it follows that $B^T\varphi(\overline{F}) + R\overline{F} = 0$, or, equivalently, $\overline{F} = -R^{-1}B^T\varphi(\overline{F})$. Substituting this in $\Phi(\overline{F}, \varphi(\overline{F})) = 0$ yields

$$Q + A^T\varphi(\overline{F}) + \varphi(\overline{F})A - \varphi(\overline{F})S\varphi(\overline{F}) = 0.$$

Furthermore, since $\overline{F} \in \mathcal{F}$, the matrix $A - S\varphi(\overline{F})$ is stable. Hence, $\varphi(\overline{F})$ is the stabilizing solution of the algebraic Riccati equation (5.7). □

Theorems 5.2.3 and 5.2.4 are combined in the following corollary.

Corollary 5.2.5 *Consider Problem 5.2.1. Define* $S := BR^{-1}B^T$. *There exists an* $\overline{F} \in \mathcal{F}$ *such that* $J(\overline{F}, x_0) \le J(F, x_0)$ *for each initial state* x_0 *and for each* $F \in \mathcal{F}$ *if and only if the algebraic Riccati equation* (5.7) *has a stabilizing solution* X. *If this condition holds, then* $\overline{F} = -R^{-1}B^T X$ *and* $J(\overline{F}, x_0) = x_0^T X x_0$.

Remark 5.2.6 If the algebraic Riccati equation (5.7) has no stabilizing solution two situation can occur. (i) The algebraic Riccati equation has a real symmetric solution X such that $A - SX$ has an eigenvalue on the imaginary axis. Then [123, Theorem 7], if (A, B) is controllable, $J(F, x_0)$ has a minimum if and only if x_0 belongs to the spectral subspace of $A - SX$ corresponding to the eigenvalues in the open left-half plane. (ii) The algebraic Riccati equation has no real symmetric solution. In that case, it is to be expected (see Theorem 5.2.2) that the infimum of $J(F, x_0)$ over $F \in \mathcal{F}$ equals $-\infty$ for all initial states x_0. As an illustration, consider the scalar case, i.e. let $A = a$, $S = s$, and $Q = q$ for some $a, s, q \in I\!R$ with $s > 0$, then the stabilizing solution of the algebraic Riccati equation (5.7) exists if and only if $a^2 + sq > 0$. It is easily seen that $J(F, x_0) = (x_0^2/s)(\xi + 2a + (a^2 + sq)/\xi)$ with $\xi = -(A + BF)$, which (i) has a minimum if $a^2 + sq > 0$, (ii) has an infimum (attained in the limit $\xi \downarrow 0$), if $a^2 + sq = 0$, and (iii) is unbounded from below if $a^2 + sq < 0$. □

5.3 Deterministic Feedback Nash Equilibria

Consider an N-player infinite-horizon stable linear time-invariant differential game, where the information structure of each player is a feedback pattern and where they are restricted to linear time-invariant strategies; see Section 2.3 for the terminology and corresponding notations. The cost function of player i is the functional $J_i : \mathcal{F}_N \times I\!R^n \to I\!R$, defined by

$$J_i(F_1, \dots, F_N, x_0) = \int_0^\infty x^T \left(Q_i + \sum_{j=1}^N F_j^T R_{ij} F_j \right) x\, dt \tag{5.11}$$

where Q_i and R_{ij} are symmetric matrices of appropriate dimensions, and $R_{ii} > 0$ for all $i = 1, \dots, N$. Consider the following definition of an equilibrium concept.

Definition 5.3.1 An N-tuple $\overline{F} = (\overline{F}_1, \dots, \overline{F}_N) \in \mathcal{F}_N$ is called a *deterministic feedback Nash equilibrium* if for all $i = 1, \dots, N$ the following inequalities hold:

$$J_i(\overline{F}, x_0) \le J_i(\overline{F}_{-i}(F_i), x_0) \tag{5.12}$$

for all initial states x_0 and for all $F_i \in I\!R^{m_i \times n}$ such that $\overline{F}_{-i}(F_i) \in \mathcal{F}_N$. □

The following theorem is a natural extension of Corollary 5.2.5 in the context of an N-player differential game.

Theorem 5.3.2 *Consider an N-player infinite-horizon stable linear time-invariant differential game, where the information structure of each player is a feedback pattern and where they are restricted to linear time-invariant strategies. Let the cost function of player i be defined by (5.11). Assume that the matrices Q_i and R_{ij} are symmetric and that $R_{ii} > 0$ for all $i = 1, \dots, N$. Define $S_i := B_i R_{ii}^{-1} B_i^T$ and $S_{ij} := B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T$ for all $i, j = 1, \dots, N$ with $i \neq j$. Then there exists a deterministic feedback Nash equilibrium if and only if there exist N real symmetric $n \times n$ matrices X_i such that*

$$Q_i + A^T X_i + X_i A - \sum_{j \neq i}^{N} (X_i S_j X_j + X_j S_j X_i) - X_i S_i X_i + \sum_{j \neq i}^{N} X_j S_{ij} X_j = 0; \tag{5.13}$$

$$A - \sum_{j=1}^{N} S_j X_j \text{ is stable.} \tag{5.14}$$

If this condition holds, the N-tuple of feedback matrices $(\overline{F}_1, \dots, \overline{F}_N)$ with

$$\overline{F}_i := -R_{ii}^{-1} B_i^T X_i \tag{5.15}$$

is a deterministic feedback Nash equilibrium and

$$J_i(\overline{F}_1, \dots, \overline{F}_N, x_0) = x_0^T X_i x_0. \tag{5.16}$$

Proof Assume that $\overline{F} = (\overline{F}_1, \dots, \overline{F}_N) \in \mathcal{F}_N$ is a deterministic feedback Nash equilibrium. Then by definition

$$J_i(\overline{F}, x_0) \leq J_i(\overline{F}_{-i}(F_i), x_0)$$

for all feedback matrices $F_i \in I\!R^{m_i \times n}$ such that $\overline{F}_{-i}(F_i) \in \mathcal{F}_N$, for all initial states x_0, and for all $i = 1, \dots, N$. Hence, according to Corollary 5.2.5, for each $i = 1, \dots, N$ there exists a real symmetric $n \times n$ matrix X_i satisfying

$$Q_i + \sum_{j \neq i}^{N} \overline{F}_j^T R_{ij} \overline{F}_j + \left(A + \sum_{j \neq i}^{N} B_j \overline{F}_j \right)^T X_i + X_i \left(A + \sum_{j \neq i}^{N} B_j \overline{F}_j \right) - X_i S_i X_i = 0 \tag{5.17}$$

such that the matrix

$$A + \sum_{j \neq i}^{N} B_j \overline{F}_j - S_i X_i \tag{5.18}$$

is stable. Corollary 5.2.5 implies moreover that $\overline{F}_i = -R_{ii}^{-1}B_i^TX_i$ and $J_i(\overline{F}, x_0) = x_0^TX_ix_0$ for all $i = 1, \ldots, N$. Replacing $\overline{F}_j$ in (5.17) by $-R_{jj}^{-1}B_j^TX_j$ for all $j \neq i$ shows that the N-tuple $(X_1, \ldots, X_N)$ of real symmetric $n \times n$ matrices satisfies the coupled set of algebraic Riccati equations (5.13). Furthermore, replacing $\overline{F}_j$ in (5.18) by $-R_{jj}^{-1}B_j^TX_j$ for all $j \neq i$ shows that the matrix $A - \sum_{j=1}^{N} S_jX_j$ is stable. This proves the "only if" part of the theorem. The "if" part can be seen in a similar way. □

The system of coupled algebraic Riccati equations (5.13) together with the stability requirement is briefly discussed in the text following Theorem 5.5.2. The same system of Riccati equations also appears in the context of feedback Nash equilibria for linear quadratic N-player differential games on the finite horizon $[0, T]$ as $T \to \infty$. Indeed, if the limit of a feedback Nash equilibrium corresponding to the solution of a system of N coupled matrix Riccati differential equations exists for $T \to \infty$, it necessarily corresponds to a solution of the algebraic Riccati equations (5.13). In fact, if the matrices Q_i and R_{ij} are positive semi-definite, it can be shown that any solution of (5.13) satisfying certain stabilizability and detectability conditions defines a Nash equilibrium in which the closed-loop system is stable [10, Corollary 6.5 and Proposition 6.8]. More details about the limiting behavior of feedback Nash equilibria in the context of finite-horizon linear quadratic differential games can be found in [99, 122]. In [99] one also finds a geometric formulation of the coupled algebraic Riccati equations (5.17).

In the framework of the present section, the restriction that feedback matrices belong to the set $\mathcal{F}_N$ is essential. Indeed, Mageirou [84] (see also [10, Example 6.6]) has shown by means of an example that there exist feedback Nash equilibria in which a player can improve unilaterally by choosing a feedback matrix for which the closed-loop system is unstable.

5.4 Stochastic Interpretation

In this section the analysis of Section 5.2 is put into a stochastic framework.

Problem 5.4.1 Consider a linear noisy system

$$\dot{x} = Ax + Bu + Ew \tag{5.19}$$

where w is a stationary white Gaussian noise with zero mean and covariance $\mathrm{E}\left(w(t)w(\tau)^T\right) = \hat{Q}\delta(t-\tau)$. The cost functional $L : \mathcal{F} \to I\!R$ is defined by

$$L(F) = \lim_{T\to\infty} \frac{1}{T}\,\mathrm{E}\int_0^T (x^TQx + u^TRu)dt, \quad u = Fx. \tag{5.20}$$

The matrices $\hat{Q}$, Q and R are symmetric and $\hat{Q} \geq 0$, $R > 0$. Determine, if it exists, a feedback matrix $\overline{F} \in \mathcal{F}$ such that $L(\overline{F}) \leq L(F)$ for all $F \in \mathcal{F}$. □

This problem is a standard infinite-horizon LQG problem. It differs from the usual setup in that the matrix Q is not necessarily positive semi-definite; see for instance [6, Section 8.2] for a standard setting. It is well-known [39, Theorem 5.4.4] that if Q is positive semi-definite, a solution of Problem 5.4.1 is given by $-R^{-1}B^TX$ with X the unique positive semi-definite solution of the algebraic Riccati equation

$$Q + A^TX + XA - XSX = 0, \quad S := BR^{-1}B^T, \tag{5.21}$$

which exists under the extra condition that (Q, A) is detectable. Note that the feedback matrix $-R^{-1}B^TX$ is independent of the matrices E and $\hat{Q}$. Here it is shown that this feedback matrix is also a solution of Problem 5.4.1 if Q is not positive semi-definite. In this case the matrix X is characterized as the stabilizing solution of (5.21). Using the results of the previous section it is moreover shown that the existence of this stabilizing solution is necessary for the existence of a solution of Problem 5.4.1 independent of the matrices E and $\hat{Q}$.

The analysis starts with the derivation of an algebraic expression for the costs $L(F)$. For this the following result is useful. This proposition can for instance be found in [6, Appendix B.13].

Proposition 5.4.2 *Consider Problem 5.4.1. Let $F \in \mathcal{F}$ and set $u = Fx$ in* (5.19). *Then*

$$\mathrm{E}\left(x(t)x(t)^T\right) = W \tag{5.22}$$

with W the unique solution of the Lyapunov equation

$$W(A + BF)^T + (A + BF)W = -E\hat{Q}E^T. \tag{5.23}$$

The following result is now straightforward.

Lemma 5.4.3 *Let $F \in \mathcal{F}$ and let P be the unique solution of the Lyapunov equation*

$$(A + BF)^TP + P(A + BF) = -Q - F^TRF. \tag{5.24}$$

Then $L(F) = \mathrm{tr}\left(E\hat{Q}E^TP\right)$.

Proof From Proposition 5.4.2 it follows that

$$\mathrm{E}\int_0^T x^T(Q+F^TRF)x\,dt = \mathrm{E}\int_0^T \mathrm{tr}\left(x^T(Q+F^TRF)x\right)dt = T\,\mathrm{tr}\left((Q+F^TRF)W\right)$$

with the matrix W determined by (5.23). Hence

$$\begin{aligned} L(F) &= \mathrm{tr}\left((Q+F^TRF)W\right) = \mathrm{tr}\left(-(A+BF)^TPW - P(A+BF)W\right) = \\ &= \mathrm{tr}\left(-W(A+BF)^TP - (A+BF)WP\right) = \mathrm{tr}\left(E\hat{Q}E^TP\right). \end{aligned}$$

□

According to the discussion following Problem 5.4.1 the following theorem provides a solution for Problem 5.4.1.

Theorem 5.4.4 *Consider Problem 5.4.1. Assume that the algebraic Riccati equation* (5.21) *has a stabilizing solution X. Define the matrix $\overline{F} := -R^{-1}B^TX$. Then $L(F) \leq L(\overline{F})$ for all $F \in \mathcal{F}$ and $L(\overline{F}) = \mathrm{tr}\left(E\hat{Q}E^TX\right)$.*

Proof It can easily be seen that

$$\begin{aligned} Q+F^TRF &= Q - XSX - F^TB^TX - XBF + (F-\overline{F})^TR(F-\overline{F}) = \\ &= -(A+BF)^TX - X(A+BF) + (F-\overline{F})^TR(F-\overline{F}). \end{aligned}$$

From the proof of Lemma 5.4.3 it is immediately clear that $L(F) = \mathrm{tr}\left((Q+F^TRF)W\right)$ where W follows from (5.23). Thus

$$\begin{aligned} L(F) &= \mathrm{tr}\left(-(A+BF)^TXW - X(A+BF)W\right) + \mathrm{tr}\left((F-\overline{F})^TR(F-\overline{F})W\right) = \\ &= \mathrm{tr}\left(E\hat{Q}E^TX\right) + \mathrm{tr}\left((F-\overline{F})^TR(F-\overline{F})W\right). \end{aligned}$$

Since $\hat{Q}$ is positive semi-definite, W is also positive semi-definite. Hence $L(\overline{F}) \leq L(F)$ for all $F \in \mathcal{F}$ and $L(\overline{F}) = \mathrm{tr}\left(E\hat{Q}E^TX\right)$. □

Consider the parameter set (A, B, Q, R) and the criterion $J : \mathcal{F} \times I\!R^n \to I\!R$ as defined by Problem 5.2.1. In the proof of Theorem 5.2.4 it has been noted that $J(F, x_0) = x_0^T\varphi(F)x_0$ where $\varphi : \mathcal{F} \to I\!R^{n\times n}$ is defined by $\varphi : F \mapsto P$ with P the solution of the Lyapunov equation (5.24). Consider for the same parameter set and the matrices E and $\hat{Q}$ the criterion $L : \mathcal{F} \to I\!R$ as defined by Problem 5.4.1. In Lemma 5.4.3 it has been shown that $J(F) = \mathrm{tr}\left(E\hat{Q}E^T\varphi(F)\right)$. A useful relation between the criteria J and L is formulated in the next lemma. This relation leads to the conclusion that the existence of the stabilizing solution of (5.21) is also necessary for the existence of a solution of Problem 5.4.1 independent of the matrices E and $\hat{Q}$.

Lemma 5.4.5 *Denote the set of all positive semi-definite $q \times q$ matrices by $\mathcal{S}_q$. The following statements are equivalent:*

(i) $\exists \overline{F} \in \mathcal{F}\, \forall E \in \mathbb{R}^{n\times q}\, \forall \hat{Q} \in \mathcal{S}_q\, \forall F \in \mathcal{F}:\ L(\overline{F}) \leq L(F)$,

(ii) $\exists \overline{F} \in \mathcal{F}\, \forall x_0 \in \mathbb{R}^n\, \forall F \in \mathcal{F}:\ J(\overline{F}, x_0) \leq J(F, x_0)$.

Proof Assume that statement (i) holds. Let $x_0 \in \mathbb{R}^n$. Choose $\hat{Q} = I$ and $E = [x_0\ \ 0 \cdots 0]$. Then $E\hat{Q}E^T = x_0x_0^T$, and thus

$$J(\overline{F}, x_0) = \operatorname{tr}\left(x_0x_0^T\varphi(\overline{F})\right) = \operatorname{tr}\left(E\hat{Q}E^T\varphi(\overline{F})\right) = L(\overline{F}) \leq L(F) = J(F, x_0).$$

Conversely, assume that statement (ii) holds. Let $E \in \mathbb{R}^{n\times q}$ and $\hat{Q} \in \mathcal{S}_q$. Since $E\hat{Q}E^T$ is positive semi-definite, there exists a matrix $Y \in \mathbb{R}^{n\times\nu}$ for some $\nu \in \mathbb{N}$ such that $E\hat{Q}E^T = YY^T$. Denote the i-th column of Y by y_i. Then

$$L(F) = \operatorname{tr}\left(YY^T\varphi(F)\right) = \operatorname{tr}\left(Y\varphi(F)Y^T\right) = \sum_{i=1}^{\nu} y_i^T\varphi(F)y_j = \sum_{i=1}^{\nu} J(F, y_i).$$

Since $J(\overline{F}, y_i) \leq J(F, y_i)$ for all $i = 1, \ldots, \nu$, it follows that

$$L(\overline{F}) = \sum_{i=1}^{\nu} J(\overline{F}, y_i) \leq \sum_{i=1}^{\nu} J(F, y_i) = L(F).$$

□

In Corollary 5.2.5 necessary and sufficient conditions are formulated for condition (ii) in Lemma 5.4.5 to hold. Furthermore, from the proof of this lemma it is clear that the feedback matrix $\overline{F} \in \mathcal{F}$ satisfies statement (i) if and only if it satisfies statement (iii). These observations lead to the following result.

Theorem 5.4.6 *Consider Problem 5.4.1. There exists an $\overline{F} \in \mathcal{F}$, independent of the matrices E and $\hat{Q}$, such that $L(\overline{F}) \leq L(F)$ for all $F \in \mathcal{F}$ if and only if the algebraic Riccati equation* (5.21) *has a stabilizing solution X. If this condition holds, then $\overline{F} = -R^{-1}B^TX$ and $L(\overline{F}) = \operatorname{tr}(E\hat{Q}E^TX)$.*

Note that Theorem 5.4.4, which has been shown independently of results from the previous section, is precisely the "if" part of Theorem 5.4.6.

5.5 Stochastic Variance-Independent Feedback Nash Equilibria

A brief survey of stochastic differential games can be found in [10, Section 6.7]. In this reference a finite-horizon framework is considered. Here a linear quadratic infinite-horizon framework is the starting point.

Consider an N-player infinite-horizon stable stochastically disturbed linear time-invariant differential game, where the information structure of each player is a feedback pattern and where they are restricted to linear time-invariant strategies; see Section 2.3 for the terminology and corresponding notation. Assume that the stochastic process is a standard white noise process. The cost function of player i is the expected value of the averaged cost per time unit, i.e. the functional $L_i : \mathcal{F}_N \to I\!R$, defined by

$$L_i(F_1, \dots, F_N) = \lim_{T\to\infty} \frac{1}{T} \,\mathrm{E} \int_0^T x^T \left(Q_i + \sum_{j=1}^{N} F_j^T R_{ij} F_j \right) x\,dt \tag{5.25}$$

where Q_i and R_{ij} are symmetric matrices of appropriate dimensions, and $R_{ii} > 0$ for all $i = 1, \dots, N$. Consider the following definition of an equilibrium concept.

Definition 5.5.1 An N-tuple $\overline{F} = (\overline{F}_1, \dots, \overline{F}_N) \in \mathcal{F}_N$ is called a *stochastic variance-independent feedback Nash equilibrium* if for all $i = 1, \dots, N$ the following inequality holds:

$$L_i(\overline{F}) \le L_i(\overline{F}_{-i}(F_i)) \tag{5.26}$$

for all matrices E and $\hat{Q}$, and for all $F_i \in I\!R^{m_i \times n}$ such that $\overline{F}_{-i}(F_i) \in \mathcal{F}_N$. □

The following theorem, which is a straightforward generalization of Theorem 5.4.6, characterizes all stochastic variance-independent feedback Nash equilibria.

Theorem 5.5.2 *Consider an N-player linear time-invariant infinite-horizon stable stochastically disturbed differential game, where the information structure of each player is a feedback pattern and where they are restricted to linear time-invariant strategies. Assume that the stochastic process is a standard white noise process. Let the cost function of player i be defined by* (5.25)*. Assume that the matrices Q_i and R_{ij} are symmetric and that $R_{ii} > 0$ for all $i = 1, \dots, N$. Define $S_i := B_i R_{ii}^{-1} B_i^T$ and $S_{ij} := B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T$ for all $i, j = 1, \dots, N$ with $i \neq j$. Then there* *exists a stochastic variance-independent feedback Nash equilibrium if and only if there exist N real symmetric $n \times n$ matrices X_i such that*

$$Q_i + A^T X_i + X_i A - \sum_{j \neq i}^{N} (X_i S_j X_j + X_j S_j X_i) - X_i S_i X_i + \sum_{j \neq i}^{N} X_j S_{ij} X_j = 0; \tag{5.27}$$

$$A - \sum_{j=1}^{N} S_j X_j \text{ is stable.} \tag{5.28}$$

If this condition holds, the N-tuple of feedback matrices $(\overline{F}_1, \dots, \overline{F}_N)$ with

$$\overline{F}_i := -R_{ii}^{-1} B_i^T X_i \tag{5.29}$$

is a stochastic variance-independent feedback Nash equilibrium and

$$L_i(\overline{F}_1, \dots, \overline{F}_N) = \operatorname{tr}\left(E\hat{Q}E^T X_i\right). \tag{5.30}$$

For a given parameter set (A, B_i, Q_i, R_{ij}), the set of stochastic variance-independent feedback Nash equilibria coincides with the set of deterministic feedback Nash equilibria (see Section 5.3). Such equilibria are completely characterized by all the real symmetric solutions $(X_1, \dots, X_N)$ of the system of coupled algebraic Riccati equations (5.27) which satisfy the condition that $A - \sum_{j=1}^{N} S_j X_j$ is stable. The system of coupled algebraic Riccati equations can be considered as a system of N matrix equations in N matrix unknowns. Alternatively, since the equations are symmetric and since the solutions are required to be real symmetric, this system can be considered as a system of $Nn(n+1)/2$ quadratic scalar equations in $Nn(n+1)/2$ real scalar unknowns. This makes the problem well-posed and one expects at most $(Nn(n+1)/2)^2$ generically different solutions. Clearly, the stability condition needs to be verified for each of these solutions. The following observation can be used if $A - \sum_{j=1}^{N} S_j X_j$ has all its eigenvalues in the open right-half plane.

Remark 5.5.3 It is easily seen that if $(X_1, \dots, X_N)$ is a real symmetric solution of the system of algebraic Riccati equations corresponding to the parameters (A, B_i, Q_i, R_{ij}) such that $A - \sum_{j=1}^{N} S_j X_j$ has all its eigenvalues in the open right-half plane, then $-(X_1, \dots, X_N)$ is a real symmetric solution of the algebraic Riccati equations corresponding the parameters $(-A, B_i, Q_i, R_{ij})$ such that $-A - \sum_{j=1}^{N} S_j X_j$ is stable. □

In the following section the system of coupled algebraic Riccati equation together with the stability requirement is studied for a simplified case.

5.6 The Two-Player Scalar Case

Consider the system of coupled algebraic Riccati equations (5.13) (or (5.27)) for scalar parameters. Assume that $N = 2$ and $R_{12} = R_{21} = 0$. Let $A = a$, $b_i = B_i$, $q_i = Q_i$, $r_i = R_{ii}$, and $s_i := b_i^2/r_i \geq 0$ for $i = 1, 2$. Recall from Theorem 5.3.2 and Theorem 5.5.2 that a pair $(x_1, x_2) \in I\!R^2$ defines a deterministic feedback Nash equilibrium $(-b_1x_1/r_1, -b_1x_2/r_2)$ or a stochastic variance-independent feedback Nash equilibrium $(-b_1x_1/r_1, -b_1x_2/r_2)$, respectively, if and only if

$$q_1 + 2ax_1 - s_1x_1^2 - 2s_2x_1x_2 = 0, \tag{5.31}$$

$$q_2 + 2ax_2 - 2s_1x_1x_2 - s_2x_2^2 = 0, \tag{5.32}$$

$$a - s_1x_1 - s_2x_2 < 0. \tag{5.33}$$

The analysis in the present section is concerned with the solvability of these three conditions. Engwerda [46] and Weeren et. al. [122] studied the solvability of these conditions in the first quadrant of the (ξ_1, ξ_2)-plane in case $q_1 \geq 0$ and $q_2 \geq 0$. Under this condition a solution (ξ_1, ξ_2) of (5.31) and (5.32) in this area automatically satisfies (5.33); see e.g. [10, Proposition 6.8]. Weeren showed that the set of equations (5.31) and (5.32) has either one or three such solutions. Engwerda found necessary and sufficient conditions under which these different situations occur. Here explicit solutions of the system (5.31)-(5.33) are derived in the special cases $s_1 = 0$ and $q_1 = 0$. Next, this system is interpreted geometrically for arbitrary values of the parameters.

The following theorem provides a complete solution of (5.31)-(5.33) if $s_1 = 0$.

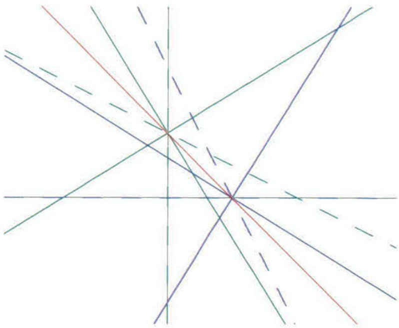
Theorem 5.6.1 *Assume that $s_1 = 0$. If $s_2 \neq 0$, there exists a solution $(x_1, x_2) \in I\!R^2$ of (5.31)-(5.33) if and only if $a^2 + s_2q_2 > 0$. If this condition holds the solution is unique and equal to*

$$\left(\frac{q_1}{2\sqrt{a^2 + s_2q_2}}, \frac{a + \sqrt{a^2 + s_2q_2}}{s_2} \right). \tag{5.34}$$

If $s_2 = 0$, there exists a solution $(x_1, x_2) \in I\!R^2$ of (5.31)-(5.33) if and only if $a < 0$. If this condition holds, the solution is unique and equal to $(-q_1/2a, -q_2/2a)$.

Proof Consider the case $s_2 \neq 0$. The equation (5.32) has the real solutions

$$x_2 = \frac{a \pm \sqrt{a^2 + s_2q_2}}{s_2} \tag{5.35}$$



green lines	:	principal axes of $\mathcal{C}_1$
green dashed lines	:	asymptotes of $\mathcal{C}_1$
blue lines	:	principal axes of $\mathcal{C}_2$
blue dashed lines	:	asymptotes of $\mathcal{C}_2$

Figure 5.1: Principal axes and asymptotes of $\mathcal{C}_1$ and $\mathcal{C}_2$.

if and only if $a^2+s_2q_2 \geq 0$. In this case $a - s_2x_2 = \mp\sqrt{a^2+s_2q_2}$. This shows that (5.33) holds if and only if the plus sign is chosen in (5.35) and $a^2+s_2q_2 > 0$. Finally, substituting the expression (5.35) with a plus sign in (5.31) yields $x_1 = q_1/2\sqrt{a^2+s_2q_2}$. The second claim of the theorem is trivial. □

A similar result can straightforwardly be derived for the case $s_2 = 0$. Assume further that $s_1 \neq 0$ and $s_2 \neq 0$, i.e. that these numbers are positive. Introduce new variables $\xi_i := s_ix_i$ and define $\sigma_i := s_iq_i$. Then it is directly seen that the system (5.31)-(5.33) is equivalent to

$$\sigma_1 + 2a\xi_1 - \xi_1^2 - 2\xi_1\xi_2 = 0, \tag{5.36}$$

$$\sigma_2 + 2a\xi_2 - 2\xi_1\xi_2 - \xi_2^2 = 0, \tag{5.37}$$

$$a - \xi_1 - \xi_2 < 0. \tag{5.38}$$

The following theorem provides a complete solution if $\sigma_1 = 0$.

Theorem 5.6.2 *Assume that $\sigma_1 = 0$. The system* (5.36)-(5.38) *has the solution*

$$\left(0, a + \sqrt{a^2+\sigma_2}\right) \tag{5.39}$$

if and only if $a^2+\sigma_2 > 0$. Furthermore, this system has the solution

$$\frac{1}{3}\left(4a - 2\sqrt{a^2+12a-3\sigma_2}, a + \sqrt{a^2+12a-3\sigma_2}\right) \tag{5.40}$$

if and only if $a > 0$ *and* $4a - a^2 < \sigma_2 \le 4a + a^2/3$. *Finally, it has the solution*

$$\frac{1}{3}\left(4a + 2\sqrt{a^2 + 12a - 3\sigma_2}, a - \sqrt{a^2 + 12a - 3\sigma_2}\right) \tag{5.41}$$

if and only if $a > 0$ *and* $\sigma_2 \le 4a + a^2/3$, *or* $a \le 0$ *and* $\sigma_2 < 4a - a^2$.

Proof Equation (5.36) allows for a distinction between two cases, i.e. $\xi_1 = 0$ and $\xi_1 = 2a - 2\xi_2$. Consider the case $\xi_1 = 0$. Then (5.37) has the solution

$$\xi_2 = a \pm \sqrt{a^2 + \sigma_2} \tag{5.42}$$

if and only if $a^2 + \sigma_2 \ge 0$. In this case $a - \xi_2 = \mp\sqrt{a^2 + \sigma_2}$. Hence, (5.38) holds if and only if the plus sign is chosen in (5.42) and $a^2 + \sigma_2 > 0$. This shows the first part of the theorem. To prove the second and third part, consider the case $\xi_1 = 2a - 2\xi_2$. Then (5.38) holds if and only if $\xi_2 < a$. Furthermore, (5.37) has the solutions

$$\xi_2 = \frac{1}{3}\left(a \pm \sqrt{a^2 + 12a - 3\sigma_2}\right) \tag{5.43}$$

if and only if $a^2 + 12a - 3\sigma_2 \ge 0$. Consider the case with a plus sign in (5.43). Then the condition $\xi_2 < a$ can only hold if $a > 0$. In this case, this condition is equivalent to the condition $\sigma_2 > 4a - a^2$. This proves the second part of the theorem. Finally, consider the case with a minus sign in (5.43). Then the condition $\xi_2 < a$ holds if $a > 0$. If $a \le 0$, this condition is equivalent to the condition $\sigma_2 < 4a - a^2$, which ensures that $\sigma_2 \le 4a + a^2/3$. □

A similar result can straightforwardly be derived in the case $\sigma_2 = 0$.

In the rest of the present section the conditions (5.36)-(5.38) are interpreted geometrically for arbitrary real numbers a, σ_1, and σ_2.

Let C_1 be the curve described by equation (5.36). This equation can be rewritten as

$$\begin{bmatrix}\xi_1 & \xi_2\end{bmatrix} A_1 \begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} + 2\begin{bmatrix}-a & 0\end{bmatrix}\begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} - \sigma_1 = 0, \quad A_1 := \begin{bmatrix}1 & 1 \\ 1 & 0\end{bmatrix}. \tag{5.44}$$

The matrix A_1 has a negative eigenvalue $-\lambda_1$ and a positive eigenvalue λ_2 with

$$\lambda_1 := \frac{1}{2}\left(\sqrt{5} - 1\right) \approx 0.618, \quad \lambda_2 := \frac{1}{2}\left(\sqrt{5} + 1\right) \approx 1.618. \tag{5.45}$$

Remark 5.6.3 The numbers λ_1 and λ_2 are known as the *golden ratio*. They satisfy some nice properties; for instance $\lambda_1 = \lambda_2 - 1 = 1/\lambda_2$. □

The curve $\mathcal{C}_1$ is thus a hyperbola, which degenerates in a pair of lines if $\sigma_1 = 0$. The principal axes and the asymptotes of this hyperbola can be found by transforming equation (5.44) in the following way. The matrix A_1 can be diagonalized as

$$A_1 = S_1 \begin{bmatrix} -\lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S_1^T, \quad S_1 := \begin{bmatrix} \beta_1 & \beta_2 \\ -\beta_2 & \beta_1 \end{bmatrix}, \tag{5.46}$$

with

$$\beta_1 := \sqrt{\frac{1}{2} - \frac{1}{10}\sqrt{5}} \approx 0.526, \quad \beta_2 := \sqrt{\frac{1}{2} + \frac{1}{10}\sqrt{5}} \approx 0.851. \tag{5.47}$$

Note that these numbers satisfy

$$\beta_1^2 + \beta_2^2 = 1, \quad \beta_2 = \lambda_2\beta_1, \quad \beta_1 = \lambda_1\beta_2. \tag{5.48}$$

Introducing the new coordinates η_1 and η_2 by

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = S_1^T \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \tag{5.49}$$

the equation (5.44) transforms into

$$-\lambda_1\eta_1^2 + \lambda_2\eta_2^2 - 2a\beta_1\eta_1 - 2a\beta_2\eta_2 - \sigma_1 = 0,$$

which can be rewritten using (5.48) as

$$-\lambda_1(\eta_1 + a\beta_2)^2 + \lambda_2(\eta_2 - a\beta_1)^2 - \sigma_1 = 0. \tag{5.50}$$

This is the equation of $\mathcal{C}_1$ on principal axes. This equation shows that the principal axes in the (η_1, η_2)-plane are described by the equations $\eta_1 = -a\beta_2$ and $\eta_2 = a\beta_1$. It is easily seen by transforming these lines back to the (ξ_1, ξ_2)-plane that the equations of the principal axis in the original coordinates are $\xi_2 = \lambda_1\xi_1 + a$ and $\xi_2 = -\lambda_2\xi_1 + a$, respectively. The asymptotes of $\mathcal{C}_1$ in the (η_1, η_2)-plane are parameterized by $\eta_1 = -a\beta_2 + t$, $\eta_2 = a\beta_1 \pm \lambda_1 t$, $t \in \mathbb{R}$. Transforming this parametrization with the plus sign back to the (ξ_1, ξ_2)-plane shows that the corresponding asymptote is described by $\xi_2 = -\xi_1/2 + a$ in the original coordinates. Similarly, the parametrization with the minus sign corresponds to the asymptote described by the equation $\xi_1 = 0$ in the original coordinates. If $\sigma_1 \geq 0$, the asymptote $\mathcal{C}_1$ intersects the principal axis $\eta_1 = -a\beta_2$ at $(-a\beta_1, a\beta_2 \pm \sqrt{\lambda_1\sigma_1})$ in the (η_1, η_2)-plane. Thus, if $\sigma_1 \geq 0$, the asymptote $\mathcal{C}_1$ intersects the principal axis $\xi_2 = \lambda_1\xi_1 + a$ at the points $(0, a) \pm \sqrt{\beta_1\beta_2\sigma_1}(1, \lambda_1)$ in the original coordinates. In a similar way, it can be seen that if $\sigma_1 \leq 0$, $\mathcal{C}_1$ intersects the other principal axis at the points $(0, a) \pm \sqrt{-\beta_1\beta_2\sigma_1}(-1, \lambda_2)$.

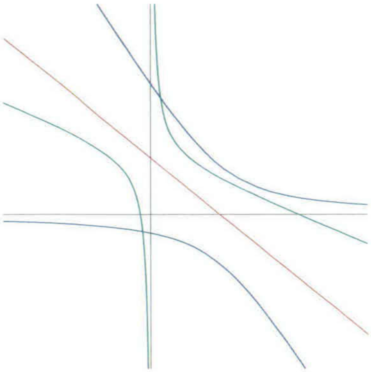


Fig. 5.2: $\sigma_1 > 0$, $\sigma_2 > 0$;
1 intersection point if $a < 0$;
1 intersection point if $a > 0$.

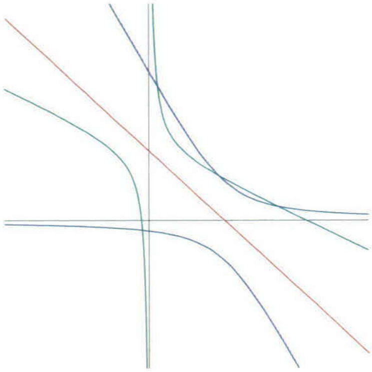


Fig. 5.3: $\sigma_1 > 0$, $\sigma_2 > 0$;
1 intersection point if $a < 0$;
3 intersection points if $a > 0$.

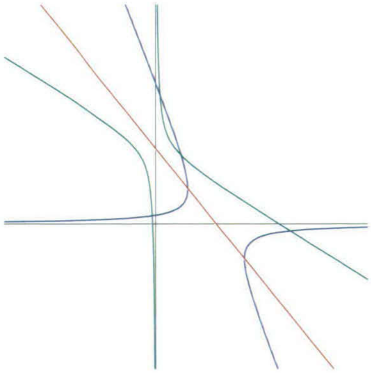


Fig. 5.4: $\sigma_1 > 0$, $\sigma_2 < 0$;
1 intersection point if $a < 0$;
3 intersection points if $a > 0$.

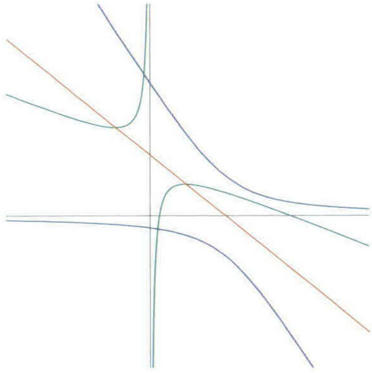


Fig. 5.5: $\sigma_1 < 0$, $\sigma_2 > 0$;
1 intersection point if $a < 0$;
1 intersection point if $a > 0$.

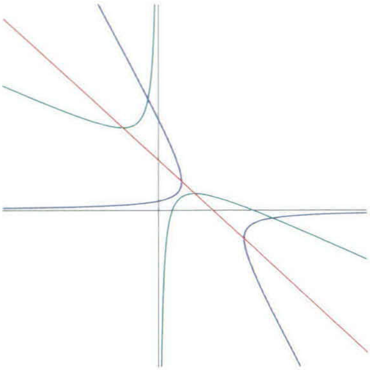


Fig. 5.6: $\sigma_1 < 0$, $\sigma_2 < 0$;
no intersection points if $a < 0$;
2 intersection points if $a > 0$.

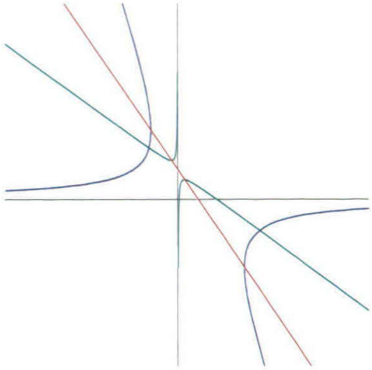


Fig. 5.7: $\sigma_1 < 0$, $\sigma_2 < 0$;
1 intersection point if $a < 0$;
1 intersection point if $a > 0$.

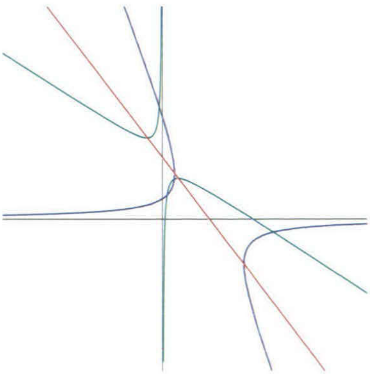


Fig. 5.8: $\sigma_1 < 0$, $\sigma_2 < 0$;
2 intersection points if $a < 0$;
2 intersection points if $a > 0$.

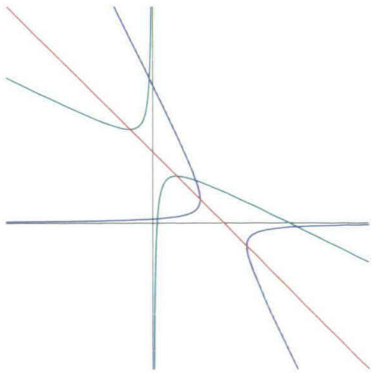


Fig. 5.9: $\sigma_1 < 0$, $\sigma_2 < 0$;
1 intersection point if $a < 0$;
3 intersection points if $a > 0$.

Let C_2 be the curve described by equation (5.37). Since the describing equations for C_1 and C_2 are essentially identical, the characteristics of C_2 can be determined in a similar way as they were determined for C_1. For instance, it is easily seen that C_2 is also a hyperbola. The precise results are summarized in Tables 5.1 and 5.2; further details concerning the derivation of the results in these tables are omitted.

	hyperbola C_1	hyperbola C_2
centre	$(0, a)$	$(a, 0)$
principal axis 1	$\xi_2 = \lambda_1\xi_1 + a$	$\xi_2 = \lambda_2(\xi_1 - a)$
principal axis 2	$\xi_2 = -\lambda_2\xi_1 + a$	$\xi_2 = -\lambda_1(\xi_1 - a)$
asymptote 1	$\xi_1 = 0$	$\xi_2 = 0$
asymptote 2	$\xi_2 = -\frac{1}{2}\xi_1 + a$	$\xi_2 = -2(\xi_1 - a)$

Table 5.1: Characteristic properties of the hyperbolas C_1 and C_2.

	$\sigma_1, \sigma_2 \geq 0$	$\sigma_1, \sigma_2 \leq 0$
principal axis 1 of C_1	$(0, a) \pm \sqrt{\beta_1\beta_2\sigma_1}(1, \lambda_1)$	
principal axis 2 of C_1		$(0, a) \pm \sqrt{-\beta_1\beta_2\sigma_1}(-1, \lambda_2)$
principal axis 1 of C_2	$(a, 0) \pm \sqrt{\beta_1\beta_2\sigma_2}(\lambda_1, 1)$	
principal axis 2 of C_2		$(a, 0) \pm \sqrt{-\beta_1\beta_2\sigma_2}(-\lambda_2, 1)$

Table 5.2: Intersection points of C_1 and C_2 with principal axes.

The position of the principal axes and asymptotes is independent of the numbers σ_1 and σ_2. They only depend on the number a. In Figure 5.1, page 91, the principal axes and asymptotes of both hyperbolas are drawn for a positive a. The $\xi_{1,2}$-axes are the two black lines; the green lines are the principal axes of C_1; the green dashed lines are the asymptotes of C_1; the blue lines are the principal axes of C_2; the blue dashed lines are the asymptotes of C_2. Note that the first asymptote of C_1 coincides with the ξ_2-axis; similarly the first asymptote of C_2 coincides with the ξ_1-axis. The red line separates the open half-planes $\xi_2 > -\xi_1 + a$ (the right side of this line) and $\xi_2 < -\xi_1 + a$ (the left side of this line). Now, a point $(\xi_1, \xi_2) \in I\!R^2$ is a solution of (5.36)-(5.38) if and only if it is an intersection point of C_1 and C_2 in the open half-plane $\xi_2 > -\xi_1 + a$. Furthermore, it is easily seen that if (ξ_1, ξ_2) is an intersection point of the two hyperbolas in the open half-plane $\xi_2 < -\xi_1 + a$, then $(-\xi_1, -\xi_2)$ is a solution of (5.36)-(5.38) with a replaced by $-a$ (see also Remark 5.5.3). Geometrically, these observations mean the following. Let the red line in Figure 5.1 correspond to the equation $\xi_2 = \xi_1 + \alpha$ for some $\alpha \geq 0$. An intersection point (ξ_1, ξ_2) of C_1 and

$\mathcal{C}_2$ situated at the right side of the red line corresponds to the case $a = \alpha \geq 0$. An intersection point situated at the left side of this line corresponds to the case $a = -\alpha < 0$. Hence the intersection points of the hyperbolas can conveniently be studied in one picture for all $a \in \mathbb{R}$.

According to Table 5.2, the characteristic position of the branches of the hyperbolas depends on the signs of σ_1 and σ_2. Furthermore, this table shows that the distance between the intersection points of the hyperbola $\mathcal{C}_i$ with the principal axes and its centre increases if the number $|\sigma_i|$ increases. In Figures 5.2-9, pages 94 and 95, the hyperbolas are drawn for eight different situations. It seems that the position of the red line is not the same in all these figures. However, this is only due to a different scaling in the ξ_1 and ξ_2 direction in each of the figures. They are all drawn for the same value of a.

Figures 5.2 and 5.3 are typical for a positive value of both σ_1 and σ_2. It is clear that for $a < 0$ there exists a unique intersection point for all positive values of σ_1 and σ_2. If $a > 0$, the number of intersection points depends on the ratio between σ_1 and σ_2. If this ratio differs relatively much from one, there exists one intersection point; if it is relatively close to one, three intersection points exist. Engwerda [46] found precise conditions under which either one or three intersection points exist in this case. Figure 5.2 shows an example with $a > 0$ and one equilibrium. In this case the value of σ_2 is much larger than the value of σ_1. Note that the equilibrium costs of player 2 are also much larger than the equilibrium costs of player 1. Figure 5.3 shows an example with $a < 0$ and three equilibria. From the equilibrium costs, it is clear that both players have a favorite equilibrium; player 1 prefers the equilibrium close to the ξ_2-axis and player 2 prefers the equilibrium close to the ξ_1-axis. The third equilibrium lies in the middle and could be interpreted as a compromise equilibrium. In this sense this situation resembles the famous game of the *battle of the sexes*.

Figures 5.4 and 5.5 display the situation if one of the numbers σ_1 or σ_2 is positive and the other is negative. If $a < 0$, there is a unique intersection point and it is geometrically clear that this holds for all such values of σ_1 and σ_2. If $a > 0$, there exists either a unique intersection point or three intersection points exist, which depends on the ratio between σ_1 and σ_2. For instance, if in Figure 5.4, σ_2 is kept fixed and σ_1 increases, the situation for $a > 0$ changes at a critical point from three to one intersection points for $a > 0$. Similarly, if in Figure 5.5, σ_1 is kept fixed and σ_2 decreases, the situation for $a > 0$ changes at a critical point from one to three intersection points.

Figures 5.6-9 are typical for a negative value of both σ_1 and σ_2. Four different situations can occur. Figure 5.6 shows a situation with no intersection points if $a < 0$ and two intersection points if $a > 0$. By keeping the value of σ_1 fixed and increasing the value of $|\sigma_2|$ the intersection point in the second quadrant moves at a critical point to the other side of the red line. In that case the situation shown in Figure 5.7 occurs: one intersection point for both cases $a < 0$ and $a > 0$. Alternatively, starting from the situation shown in Figure 5.6 and keeping σ_1 fixed and decreasing $|\sigma_2|$, two extra intersection points appear at a critical point. First, the situation shown in Figure 5.8 occurs: two intersection points for both cases $a < 0$ and $a > 0$. By further decreasing the value of $|\sigma_2|$, the distance between the two middle intersection points increases. At a critical point the situation changes to the situation shown in Figure 5.9: one intersection point if $a < 0$ and three intersection points if $a > 0$.

	$a < 0$	$a > 0$
$\sigma_1 > 0, \sigma_2 > 0$	1	1 or 3
$\sigma_1 > 0, \sigma_2 < 0$	1	1 or 3
$\sigma_1 < 0, \sigma_2 > 0$	1	1 or 3
$\sigma_1 < 0, \sigma_2 < 0$	0, 1, or 2	1, 2, or 3

Table 5.3: Number of intersection points of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Summarizing, the possible number of intersection points is shown in Table 5.3. It is geometrically clear that no other situations can occur generically (situations in which the hyperbolas $\mathcal{C}_1$ and $\mathcal{C}_2$ touch are excluded). The fact that there exist situations with no intersection points is not surprising. Indeed, also in the scalar one-player case the cost function is not bounded from below if q is negative and its absolute value is sufficiently large (see Remark 5.2.6). In the scalar two-player case however, if only one player has a negative state weight, a deterministic (or stochastic variance-independent) feedback Nash equilibrium always exists. If both players have a negative state weight there exist situations in which no equilibrium exists.

5.7 Concluding Remarks

Section 8.1.3 contains concluding remarks about the feedback Nash equilibrium concept introduced in Section 5.3 and the corresponding analysis of the two-player scalar case. The relation between the stochastic variance-independent feedback Nash equilibrium concept and other robust equilibria concepts introduced in this thesis is briefly mentioned in Section 8.1.4. Concluding remarks about Section 5.2 are given in Section 8.2.1. A preliminary version of this chapter has been presented at the *Ninth International Symposium on Dynamic Games and Applications* [26]. Sections 5.2-5.5 have been submitted for publication [25].

Chapter 6

Robust Equilibria with Bounded Disturbances

6.1 Introduction

In many well-developed areas in control theory uncertainty is expressed through an unknown function, the disturbance, influencing the system; sometimes people refer to this as signal uncertainty. Examples of such areas are: (i) decoupling theory [15, 118, 94], (ii) H_2 optimal control theory [118, 128], and (iii) H_∞ optimal control theory [9, 42, 50, 53, 57, 118, 128]. The design objective in these areas is: (i) to decouple the output from the disturbance, (ii) to minimize the H_2 (or (iii) H_∞) norm of the transfer function from disturbance to output. The H_2 and H_∞ minimization problems have the interpretation of minimizing the worst-case effect of the disturbance, measured in a specific way. Modeling uncertainty as such does not seem to have received much attention so far in the context of nonzero-sum dynamic games. In this thesis, a dynamic game version of a decoupling problem has been considered in Chapter 4. The theory of H_2 optimal control problems, essentially identical to LQG control, has been studied in a nonzero-sum dynamic game setting in the previous chapter. The H_∞ optimal control problem is strongly related to the soft-constrained differential game [9]. A nonzero-sum dynamic game setting with soft-constrained criteria will be developed in the next chapter. In the present chapter we assume that the disturbance is an $L_2(0, \infty)$ function, the norm of which is bounded by a given number. A dynamic game setting in which it is the objective of the players to minimize the worst-case effect of such a disturbance is developed in Section 6.4. First the zero-player and one-player cases are studied in Sections 6.2 and 6.3, respectively.

In Section 6.2 we study the problem of maximizing the L_2-norm of the output of a stable linear time-invariant closed-loop system, over all L_2-disturbances with norm bounded by a given number. This maximization problem can conveniently be generally reformulated in a Hilbert space context: given two Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, a bounded linear operator T from $\mathcal{X}$ to $\mathcal{Y}$, a fixed element $y_0 \in \mathcal{Y}$, and a number $r > 0$, maximize $\|y_0 + Tx\|$ subject to $\|x\| \leq r$. Both an analysis explicitly in terms of the system matrices (Subsection 6.2.1) and an analysis in the context of the general Hilbert space formulation (Subsection 6.2.2) are included. Under a weak condition, we show that a unique maximizing disturbance exists. By means of an example it is shown that when this condition is violated, such a disturbance may not exist.

The one-player case is studied in Section 6.3. The purpose of that section is to find a linear time-invariant internally stabilizing feedback map that minimizes the supremum of the L_2-norm of the output of a linear time-invariant system. Here the supremum is taken over all L_2-disturbances with norm bounded by a given number. For $x_0 = 0$, this minmax problem reduces to the state feedback H_∞ control problem for linear systems. For nonzero unknown initial state, the corresponding maxmin problem has been studied by Chen [32]. Here we study the minmax problem with known $x_0 \neq 0$, i.e. we study the worst-case disturbance attenuation problem with known nonzero initial state. The solution of this problem for zero initial state can be found from the corresponding infinite-horizon soft-constrained differential game in which both the controller and disturbance have a closed-loop perfect state information pattern; see Başar and Bernhard [9, Section 4.4]. They consider the soft-constrained differential game without taking $x_0 = 0$, and show that a saddle-point equilibrium for such a game exists under certain conditions. The solution of the disturbance attenuation problem with zero initial state is found from this equilibrium by assuming $x_0 = 0$. In Section 6.3 we consider the corresponding soft-constrained differential game with a memoryless perfect state information pattern for the controller and an open-loop information pattern for the disturbance. It will be shown that also under such an information structure a saddle-point equilibrium exists and that this solution can be used to solve the disturbance attenuation problem with nonzero initial state.

An extension of the disturbance attenuation problem to systems with an unknown nonzero initial state has already been developed by Khargonekar et al. [66] (see also [9, Section 4.5.2]). They model the initial-state uncertainty in the performance criterion. In contrast, we assume that the initial state is known to the controller designer. Therefore, a feedback map solving the problem under consideration depends in general on x_0. In applications where initial states may deviate substantially from zero, such feedback controllers could be interesting, since they specifically take this initial state into account. One could also replace the parameter x_0 in the feedback map by the current state $x(t)$, producing a nonlinear state feedback controller.

The general N-player case is studied in Section 6.4. Since the one-player solution from Section 6.3 depends on the initial state, we shall consider dynamic games where the information structure of the players is a memoryless perfect state pattern. Instead of one uniform bound on the disturbance, the players are allowed to use different bounds, i.e. each player can model his own belief about the size of the disturbance. We shall call these bounds risk-sensitivity parameters. Given an N-tuple of feedback matrices and an initial state, player i's cost function is the supremum of the L_2-norm of his output, where the supremum is taken over all disturbances less than or equal to his risk-sensitivity parameter. We shall call an N-tuple of feedback matrices a risk-sensitive memoryless perfect state Nash equilibrium if the usual equilibrium inequalities with respect to these costs are met. We derive a set of sufficient conditions in terms of a system of coupled symmetric algebraic Riccati equations, which involves endogenously determined parameters. These parameters need to be determined from a set of nonlinear equations. The two-player scalar case, which we work out in more detail, shows the difficulties of the set of sufficient conditions.

6.2 Bounded Worst-Case Disturbances

In a risk-sensitive Nash equilibrium, which will be defined in Section 6.4, the players deal with the uncertainty by minimizing the worst-case effect of bounded disturbances. In this equilibrium the player's strategies are required to stabilize the closed-loop system. In this section we study the existence and uniqueness of bounded worst-case disturbances for a given stable closed-loop system. More precisely, we consider a linear time-invariant system

$$\dot{x} = Ax + Ew, \quad x(0) = x_0, \quad z = Cx \tag{6.1}$$

with A stable. The norm of the disturbance $w \in L_2^q(0,\infty)$ is assumed to be bounded by a given number $r > 0$. We are interested in disturbances that maximize the norm of the output, i.e. we will study the following problem.

Problem 6.2.1 Let $r > 0$. Determine

$$\sup_{\|w\| \le r} \|z\|^2. \tag{6.2}$$

If this supremum is a maximum, determine the disturbances attaining this maximum. Such disturbances are called *bounded worst-case disturbances*. □

We introduce the following notations. For each $x_0 \in I\!R^n$, let $z_{x_0} \in L_2^p(0,\infty)$ be defined by

$$z_{x_0}(t) = Ce^{tA}x_0. \tag{6.3}$$

The operator $\mathcal{G} : L_2^q(0,\infty) \to L_2^p(0,\infty)$ is defined by

$$(\mathcal{G}w)(t) = C\int_0^t e^{(t-\tau)A}Ew(\tau)d\tau. \tag{6.4}$$

This is a convolution operator with kernel function $Ce^{sA}E$ for $s > 0$, and 0 for $s < 0$. In [54, Section XII.1] it is shown that such an operator is a well-defined bounded linear operator if the kernel function is integrable on the real line. Since A is stable, the kernel function here is clearly integrable on the real line. Hence, the operator $\mathcal{G}$ is a bounded linear operator. The adjoint operator of $\mathcal{G}$ is denoted by $\mathcal{G}^*$. Furthermore, let

$$\gamma^* := \|\mathcal{G}\| = \sup_{w\in L_2^q(0,\infty),\ w\neq 0} \frac{\|\mathcal{G}w\|}{\|w\|}, \tag{6.5}$$

$$J(w, x_0) := \|z\|^2, \tag{6.6}$$

$$B_r := \{w \in L_2^q(0,\infty) |\ \|w\| \leq r\}, \quad r > 0. \tag{6.7}$$

With these notations the output $z \in L_2^p(0,\infty)$ can equivalently be written as the affine image

$$z = z_{x_0} + \mathcal{G}w. \tag{6.8}$$

Note that the boundedness of $\mathcal{G}$ ensures that Problem 6.2.1 is well-defined, i.e. the supremum (6.2) exists for each positive number r.

Remark 6.2.2 Closely related to Problem 6.2.1 is the problem of minimizing $\|z\|^2$ subject to the constraint $\|w\| \leq r$. This problem has been studied by Bellman [18, Chapter 9]; see also [19, Chapter 4]. He showed that if $\mathcal{G}^* z_{x_0} \neq 0$ this minimization problem has a solution, i.e. a minimizing function exists. This in contrast to the maximization problem. Indeed, at the end of Section 6.2.2 we present an example, i.e. Example 6.2.24, in which a bounded worst-case disturbance does not exist. □

Remark 6.2.3 Problem 6.2.1 has also been studied by Chen [32, Section II]. His analysis serves as preliminary work to investigate a related maxmin problem (see also the introduction of the present chapter). His analysis of the maximization problem consists of (i) a formulation of the corresponding finite-horizon version of the problem in a Hilbert space context, (ii) the solution of this problem in terms of the finite-horizon equivalents of the operator $\mathcal{G}$, its adjoint $\mathcal{G}^*$, and the vector z_{x_0} (Lemma 1), and (iii) the remark that this lemma also holds for the infinite-horizon version of the problem (as formulated here) under the condition that A is stable. We shall present a similar result as Chen's Lemma 1 in more general terms in Theorem 6.2.18. □

Remark 6.2.4 Problem 6.2.1 includes the problem of the determination of the H_∞ norm of the system (6.1). Indeed, for $x_0 = 0$ and $r = 1$, the supremum (6.2) equals γ^{*2}, i.e. the square of the L_2-induced operator norm of $\mathcal{G}$; see (6.5). It is well-known that this norm equals the H_∞ norm of the system (6.1). There exist good methods to determine γ^* numerically; see e.g. [22, Section 5.6.3]. □

Problem 6.2.1 can be interpreted as follows. Determine the maximal norm of all the outputs that are contained in the image of B_r under the affine transformation $w \mapsto z_{x_0} + \mathcal{G}w$. It is to be expected that bounded worst-case disturbances have norm equal to r. This is true under the condition that $\mathcal{G}$ is not the zero operator, as shown by the next lemma.

Lemma 6.2.5 *Assume that $\mathcal{G} \neq 0$. Let $r > 0$, $x_0 \in \mathbb{R}^n$, and $\overline{w} \in B_r$. If $J(w, x_0) \leq J(\overline{w}, x_0)$ for all $w \in B_r$, then $\|\overline{w}\| = r$.*

Proof Consider for each $w \in L_2^q(0,\infty)$, the quadratic function $\varphi_w : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_w(\lambda) = J(\lambda w, x_0) = \|\mathcal{G}w\|^2\lambda^2 + 2\langle \mathcal{G}w, z_{x_0}\rangle \lambda + \|z_{x_0}\|^2.$$

Suppose $\mathcal{G}\overline{w} = 0$. Let $w \in L_2^q(0,\infty) \setminus \{0\}$. Since $J(w, x_0) \leq J(\overline{w}, x_0)$ if $\|w\| \leq r$, we have $\varphi_w(\lambda) \leq J(\overline{w}, x_0) = \|z_{x_0}\|^2$ for all $\lambda \in [-r/\|w\|, r/\|w\|]$, or, equivalently, $\|\mathcal{G}w\|^2\lambda^2 + 2\langle \mathcal{G}w, z_{x_0}\rangle \lambda \leq 0$ for all $\lambda \in [-r/\|w\|, r/\|w\|]$. Since $\|\mathcal{G}w\|^2 \geq 0$, this is only possible if $\mathcal{G}w = 0$. Hence, we obtain $\mathcal{G} = 0$, which is an excluded case. Without loss of generality we may therefore assume that $\mathcal{G}\overline{w} \neq 0$. Then also $\overline{w} \neq 0$. On the one hand, since $J(w, x_0) \leq J(\overline{w}, x_0)$ for all $w \in B_r$, we know that $\varphi_{\overline{w}}(\lambda)$ restricted to the interval $[-r/\|\overline{w}\|, r/\|\overline{w}\|]$ attains its maximal value at $\lambda = 1$. On the other hand, due to the quadratic structure of $\varphi_{\overline{w}}(\lambda)$ and the fact that $\mathcal{G}\overline{w} \neq 0$, the maximal value of $\varphi_{\overline{w}}(\lambda)$ restricted to the interval $[-r/\|\overline{w}\|, r/\|\overline{w}\|]$ is attained at either $\lambda = -r/\|\overline{w}\|$ or $\lambda = r/\|\overline{w}\|$. Hence, $1 = \pm r/\|\overline{w}\|$. Obviously, the minus sign is not possible, so $\|\overline{w}\| = r$, which completes the proof. □

Excluding the trivial case $\mathcal{G} = 0$, this lemma states that if a bounded worst-case disturbance exists, its norm is equal to r and hence, Problem 6.2.1 is a maximization problem under an equality constraint. Such a problem can be solved by introducing a Lagrange multiplier. For the problem under consideration, this yields a set of necessary and sufficient conditions for a bounded worst-case disturbance to exist. These conditions are formulated in the next theorem.

Theorem 6.2.6 *Consider Problem 6.2.1. Let $\mathcal{G} \neq 0$. Then, $\overline{w} \in B_r$ is a bounded worst-case disturbance if and only if (i) $\|\overline{w}\| = r$ and (ii) there exists a number $\lambda \leq -\gamma^{*2}$ such that $\overline{w}$*

maximizes the functional

$$J_\lambda(w, x_0) = J(w, x_0) + \lambda \|w\|^2 \tag{6.9}$$

in the unconstrained space $L_2^q(0, \infty)$.

Proof The sufficiency is easily seen. Indeed, let $w \in B_r$ then

$$J(w, x_0) = J_\lambda(w, x_0) - \lambda \|w\|^2 \leq J_\lambda(\overline{w}, x_0) - \lambda r^2 = J(\overline{w}, x_0).$$

Next, we prove the necessity. For that purpose, let $\overline{w} \in B_r$ maximize $J(w, x_0)$ under the constraint $w \in B_r$. Since $\mathcal{G} \neq 0$, it follows immediately from Lemma 6.2.5 that $\|\overline{w}\| = r$. The Lagrange multiplier rule (see for instance [81, Theorem 2, Section 7.7]) guarantees the existence of a number $\lambda \in I\!R$ such that the functional $J_\lambda(w, x_0)$ is stationary at $w = \overline{w}$. This statement holds under the conditions that the functionals $w \mapsto J(w, x_0)$ and $w \mapsto \|w\|^2$ are continuous and Fréchet differentiable. These conditions clearly hold. Furthermore, the derivative of $w \mapsto \|w\|^2$ must be nonzero at $\overline{w}$. In fact, it is easily seen that this derivative is nonzero at all points $w \neq 0$, so in particular for all points with norm equal to r. Since $\overline{w}$ is a stationary point of $J_\lambda(w, x_0)$, we have $\delta_1 J_\lambda(\overline{w}, x_0; \Delta w) = 0$ for all increments Δw (for the notation see Section 2.1). This differential can easily be computed. Indeed,

$$\delta_1 J_\lambda(w, x_0; \Delta w) = \lim_{\alpha \to 0} \frac{J_\lambda(w + \alpha \Delta w, x_0) - J_\lambda(w, x_0)}{\alpha} = 2 \left\langle \mathcal{G}^* z_{x_0} + \mathcal{G}^* \mathcal{G} w + \lambda w, \Delta w \right\rangle.$$

Hence, $\overline{w}$ satisfies the equation

$$(\mathcal{G}^* \mathcal{G} + \lambda) \overline{w} = -\mathcal{G}^* z_{x_0}.$$

Thus a completion of the square shows that $J_\lambda(w, x_0)$ can be rewritten as

$$J_\lambda(w, x_0) = \langle w - \overline{w}, (\mathcal{G}^* \mathcal{G} + \lambda)(w - \overline{w}) \rangle + \langle \overline{w}, \mathcal{G}^* z_{x_0} \rangle + \langle z_{x_0}, z_{x_0} \rangle. \tag{6.10}$$

Now, suppose that $\lambda > -\gamma^{*2}$. From the definition of γ^* it follows that there exists an open set $\Omega \subset L_2^q(0, \infty)$ such that $-\lambda < \|\mathcal{G} w\|^2 / \|w\|^2$ for all $w \in \Omega$. Equivalently,

$$\langle w, (\mathcal{G}^* \mathcal{G} + \lambda) w \rangle > 0 \text{ for all } w \in \Omega.$$

Thus also

$$\langle \mu w, (\mathcal{G}^* \mathcal{G} + \lambda) \mu w \rangle > 0 \text{ for all } w \in \Omega \text{ and for all } \mu \neq 0.$$

Together with (6.10) this shows that

$$J_\lambda(\overline{w} + \mu w, x_0) > J_\lambda(\overline{w}, x_0) \text{ for all } w \in \Omega \text{ and for all } \mu \neq 0.$$

Since Ω is open, there exists a $w_0 \in \Omega$ with $\langle w_0, \overline{w} \rangle \neq 0$. Consequently, due to the quadratic structure of

$$\|\overline{w} + \mu w_0\|^2 = r^2 + 2\mu \langle \overline{w}, w_0 \rangle + \mu^2 \|w_0^2\|^2$$

there exists a $\mu_0 \neq 0$ such that $\|\overline{w} + \mu_0 w_0\|^2 = r^2$. Hence we have found a $w_1 \in L_2^q(0, \infty)$, i.e. $w_1 := \overline{w} + \mu_0 w_0$, with $\|w_1\| = r$ such that $J_\lambda(w_1, x_0) > J_\lambda(\overline{w}, x_0)$. Since also $\|\overline{w}\| = r$, this is equivalent to $J(w_1, x_0) > J(\overline{w}, x_0)$. This clearly contradicts the assumption that $\overline{w}$ maximizes $J(w, x_0)$ under the constraint $w \in B_r$. Hence $\lambda \leq -\gamma^{*2}$. There remains to be shown that $\overline{w}$ maximizes $J_\lambda(w, x_0)$ in the unconstrained space $L_2^q(0, \infty)$. This follows from (6.10) by observing that the inequality $\lambda \leq -\gamma^{*2}$ implies that the operator $\mathcal{G}^*\mathcal{G} + \lambda$ is negative semi-definite. □

Motivated by this result we tackle Problem 6.2.1 in the following way. With a small abuse of notation, define the functional

$$J_\gamma(w, x_0) = J(w, x_0) - \gamma^2 \|w\|^2. \tag{6.11}$$

For each $\gamma \geq \gamma^*$, we look for disturbances $\overline{w}_\gamma$ which maximize $J_\gamma(w, x_0)$ in the unconstrained space $L_2^q(0, \infty)$. Next, we investigate whether a number $\gamma \geq \gamma^*$ exists such that $\|\overline{w}_\gamma\| = r$. Necessary and sufficient conditions for $J_\gamma(w, x_0)$ to have a maximum are formulated in the following lemma.

Lemma 6.2.7 *Let $x_0 \in \mathbb{R}^n$. The criterion $J_\gamma(w, x_0)$ has a maximum if and only if $\gamma \geq \gamma^*$ and $\mathcal{G}^* z_{x_0} \in \operatorname{im}(\mathcal{G}^*\mathcal{G} - \gamma^2)$.*

Proof The criterion $J_\gamma(w, x_0)$ can be written as

$$\begin{aligned} J_\gamma(w, x_0) &= \|z_{x_0} + \mathcal{G}w\|^2 - \gamma^2 \|w\|^2 = \\ &= \langle z_{x_0} + \mathcal{G}w, z_{x_0} + \mathcal{G}w \rangle - \gamma^2 \langle w, w \rangle = \\ &= \Big\langle w, (\mathcal{G}^*\mathcal{G} - \gamma^2) w \Big\rangle + 2 \langle w, \mathcal{G}^* z_{x_0} \rangle + \langle z_{x_0}, z_{x_0} \rangle . \end{aligned}$$

According to Theorem 2.1.1, this quadratic expression has a maximum if and only if the operator $\gamma^2 - \mathcal{G}^*\mathcal{G}$ is positive semi-definite and $\mathcal{G}^* z_{x_0} \in \operatorname{im}(\mathcal{G}^*\mathcal{G} - \gamma^2)$. Furthermore, $\gamma^2 - \mathcal{G}^*\mathcal{G}$ is positive semi-definite if and only if $\langle \mathcal{G}^*\mathcal{G}w, w \rangle \leq \gamma^2 \langle w, w \rangle$ for all $w \in L_2^q(0, \infty)$. This holds if and only if $\gamma \geq \gamma^*$. □

At this point, the analysis can be carried out further in two ways. The criterion $J_\gamma(w, x_0)$ can be formulated explicitly in terms of the matrices A, C, E, and the initial state x_0. Using manipulations in terms of this data, one can determine a unique maximizing disturbance of the criterion $J_\gamma(w, x_0)$ for each $\gamma > \gamma^*$. Under a certain condition it can be shown that γ can always be chosen such as to make the norm of this maximizing disturbance equal to r. This approach is worked out in the next subsection, i.e. in Subsection 6.2.1.

Alternatively, Problem 6.2.1 can be formulated in more general geometrical terms. With this general formulation the existence of a disturbance maximizing the criterion $J_{\gamma^*}(w, x_0)$ can be studied more conveniently than with the approach of the next subsection. This existence problem is useful to obtain more precise results about the existence and uniqueness of bounded worst-case disturbances. This approach is worked out in Subsection 6.2.2, which can be read independently from the next subsection.

6.2.1 A Matrix Approach

In the analysis preceding this subsection it has been shown that a disturbance is a bounded worst-case disturbance if and only if it has norm equal to r and if it maximizes the functional $J_\gamma(w, x_0)$ for some $\gamma \geq \gamma^*$. In the first part of the present subsection (until Remark 6.2.10) we determine for each $\gamma > \gamma^*$ a disturbance w_γ which maximizes $J_\gamma(w, x_0)$ uniquely. The second part of this subsection (after Remark 6.2.10) addresses the question whether a $\gamma > \gamma^*$ can be chosen such as to make the norm of w_γ equal to r.

We have

$$J_\gamma(w, x_0) = \int_0^\infty \left(x^T C^T C x - \gamma^2 w^T w\right) dt, \tag{6.12}$$

where x follows from $\dot{x} = Ax + Ew$, $x(0) = x_0$. The maximal value of $J_\gamma(w, x_0)$ can be determined by adding and subtracting the term $\frac{d}{dt} x^T X x$ to the integrand and subsequently, a completion of the squares. Here X denotes an arbitrary symmetric $n \times n$ matrix. Thus

$$\begin{aligned} J_\gamma(w, x_0) &= \int_0^\infty \left(x^T C^T C x - \gamma^2 w^T w + \frac{d}{dt} x^T X x - \frac{d}{dt} x^T X x\right) dt = \\ &= x_0^T X x_0 + \int_0^\infty \left(x^T (C^T C + A^T X + XA) x + 2 w^T E^T X x - \gamma^2 w^T w\right) dt, \end{aligned} \tag{6.13}$$

which holds because $x(t) \to 0$ for $t \to \infty$ (for details see Lemma 2.1.2). Completing the squares in the expression $-\gamma^2 w^T w + 2w^T E^T X x$ yields

$$-\gamma^2 w^T w + 2w^T E^T X x = -\gamma^2 \left|w - \gamma^{-2} E^T X x\right|^2 + \gamma^{-2} x^T X E E^T X x.$$

Substituting this in (6.13) produces

$$\begin{aligned} J_\gamma(w, x_0) &= x_0^T X x_0 + \\ &= \int_0^\infty \left(x^T (C^T C + A^T X + XA + \gamma^{-2} X E E^T X) x - \gamma^2 \left| w - \gamma^{-2} E^T X x \right|^2 \right) dt. \end{aligned}$$

Hence, if X satisfies the algebraic Riccati equation

$$C^T C + A^T X + XA + \gamma^{-2} X E E^T X = 0, \tag{6.14}$$

then

$$J_\gamma(w, x_0) = x_0^T X x_0 - \gamma^2 \int_0^\infty \left| w - \gamma^{-2} E^T X x \right|^2 dt. \tag{6.15}$$

Now, define

$$w_\gamma(t) := \gamma^{-2} E^T X e^{t(A + \gamma^{-2} E E^T X)} x_0. \tag{6.16}$$

A sufficient condition for w_γ to be in $L_2^q(0, \infty)$ is stability of the matrix $A + \gamma^{-2} E E^T X$. This condition is satisfied if X is the stabilizing solution of the algebraic Riccati equation (6.14). The following result gives a necessary and sufficient condition for this stabilizing solution to exist.

Lemma 6.2.8 [128, Corollary 13.24 (Bounded Real Lemma)] *The stabilizing solution of the algebraic Riccati equation* (6.14) *exists if and only if γ is greater than the H_∞ norm of the system $\dot{x} = Ax + Ew$, $z = Cx$, i.e. if and only if $\gamma > \gamma^*$.*

The following theorem is now straightforward.

Theorem 6.2.9 *Consider the linear system* (6.1) *with A stable, and denote its H_∞ norm by γ^*. For each $\gamma > \gamma^*$, the criterion $J_\gamma(w, x_0)$, as defined by* (6.11), *is uniquely maximized by the disturbance w_γ, as defined by* (6.16), *where X is the stabilizing solution of the algebraic Riccati equation* (6.14). *Furthermore $J_\gamma(w_\gamma, x_0) = x_0^T X x_0$.*

Proof Since $\gamma > \gamma^*$, it follows from the bounded real lemma that the stabilizing solution of (6.14) exists. Hence, (6.15) holds and w_γ, as defined by (6.16), is in $L_2^q(0,\infty)$. Now, (6.15) shows that $J_\gamma(w, x_0) \leq x_0^T X x_0$ for all w. Furthermore, if $J_\gamma(w, x_0) = x_0^T X x_0$, (6.15) implies $w = \gamma^{-2} E^T X x$. Thus $\dot{x} = (A + \gamma^{-2} E E^T X)x$, or, $x(t) = e^{t(A+\gamma^{-2}EE^TX)}x_0$. Consequently, $w = w_\gamma$, which completes the proof. □

Remark 6.2.10 Lemma 6.2.8 shows that the stabilizing solution of the algebraic Riccati equation does not exist if $\gamma = \gamma^*$. Therefore, the method by which Theorem 6.2.9 has been derived fails in this case. However, Lemma 6.2.7 shows that $J_{\gamma^*}(w, x_0)$ may have a maximum, depending on whether or not $\mathcal{G}^* z_{x_0} \in \operatorname{im}(\mathcal{G}^*\mathcal{G} - \gamma^{*2})$. This is not further investigated in this subsection. More details about this case can be found in Subsection 6.2.2. □

Next, we investigate the existence of a number $\gamma > \gamma^*$ such that $\|w_\gamma\| = r$. The following lemma states that $\|w_\gamma\|$ decreases if γ increases.

Lemma 6.2.11 *Let $\gamma^* < \gamma_1 < \gamma_2$. Then $\|w_{\gamma_1}\| \geq \|w_{\gamma_2}\|$.*

Proof Recall from Theorem 6.2.9 that

$$J_\gamma(w, x_0) \leq J_\gamma(w_\gamma, x_0) \text{ for all } w \in L_2^q(0,\infty) \text{ and for all } \gamma > \gamma^*. \tag{6.17}$$

Consecutively substituting $w = w_{\gamma_2}$, $\gamma = \gamma_1$ and $w = w_{\gamma_1}$, $\gamma = \gamma_2$ yields

$$\|z_{x_0} + \mathcal{G}w_{\gamma_2}\|^2 - \gamma_1^2\|w_{\gamma_2}\|^2 \leq \|z_{x_0} + \mathcal{G}w_{\gamma_1}\|^2 - \gamma_1^2\|w_{\gamma_1}\|^2, \tag{6.18}$$

$$\|z_{x_0} + \mathcal{G}w_{\gamma_1}\|^2 - \gamma_2^2\|w_{\gamma_1}\|^2 \leq \|z_{x_0} + \mathcal{G}w_{\gamma_2}\|^2 - \gamma_2^2\|w_{\gamma_2}\|^2, \tag{6.19}$$

respectively. By adding these two inequalities we obtain

$$-\gamma_1^2\|w_{\gamma_2}\|^2 - \gamma_2^2\|w_{\gamma_1}\|^2 \leq -\gamma_1^2\|w_{\gamma_1}\|^2 - \gamma_2^2\|w_{\gamma_2}\|^2,$$

or, equivalently,

$$\left(\gamma_1^2 - \gamma_2^2\right)\left(\|w_{\gamma_1}\|^2 - \|w_{\gamma_2}\|^2\right) \leq 0.$$

Now, since $\gamma_1 < \gamma_2$, this clearly implies that $\|w_{\gamma_1}\| \geq \|w_{\gamma_2}\|$. □

Remark 6.2.12 Recall from Theorem 6.2.9 that the inequality (6.17) is strict if $w \neq w_\gamma$. This implies that (6.18) and (6.19) are strict if $w_{\gamma_1} \neq w_{\gamma_2}$. Hence, if $\gamma \mapsto w_\gamma$ is injective, $\gamma \mapsto \|w_\gamma\|$ is also injective. □

The norm $\|w_\gamma\|$ decreases to zero as γ tends to infinity. Indeed, due to the stability of A, we see from the algebraic Riccati equation (6.14) that its stabilizing solution X converges to P as $\gamma \to \infty$, where P is the unique solution of the Lyapunov equation

$$A^T P + PA = -C^T C.$$

Consequently, $\gamma^2 w_\gamma(t) \to E^T P e^{tA} x_0$ for $\gamma \to \infty$. Hence, $\|w_\gamma\| \downarrow 0$ for $\gamma \to \infty$.

The behavior of $\|w_\gamma\|$ for $\gamma \downarrow \gamma^*$ is more complicated. The stabilizing solution X of the algebraic Riccati equation (6.14) does not exist if $\gamma = \gamma^*$. The definition of the stabilizing solution (see Section 2.2) states that $\operatorname{im}\,[I \;\; X]^T$ equals the spectral subspace corresponding to the eigenvalues in the open left-half plane of the Hamiltonian matrix

$$H_\gamma := \begin{bmatrix} A & \gamma^{-2} E E^T \\ -C^T C & -A^T \end{bmatrix}.$$

If $\gamma = \gamma^*$, two things can go wrong. Either the spectral subspace corresponding to the eigenvalues of H_{γ^*} in the open left-half plane is not complementary to the subspace $\operatorname{im}\,[0 \;\; I]^T$, or H_{γ^*} has eigenvalues on the imaginary axis. In the first case the spectral radius of X tends to infinity for $\gamma \downarrow \gamma^*$, and in the second case at least one of the eigenvalues of $A + \gamma^{-2} E E^T X$ approaches the imaginary axis for $\gamma \downarrow \gamma^*$. In both cases it is to be expected that $\|w_\gamma\| \to \infty$ typically. We shall not go into the problem of finding exact conditions under which this happens.

We have derived the following theorem, which solves Problem 6.2.1 under a certain condition.

Theorem 6.2.13 *Consider the linear system* (6.1) *with A stable and denote its H_∞ norm by γ^*. Consider Problem 6.2.1. Define $w_\gamma \in L_2^q(0, \infty)$ by*

$$w_\gamma(t) = \gamma^{-2} E^T X_\gamma e^{t(A + \gamma^{-2} E E^T X_\gamma)} x_0,$$

where X_γ is the stabilizing solution of the algebraic Riccati equation

$$C^T C + A^T X + XA + \gamma^{-2} X E E^T X = 0.$$

For each positive number $r < \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$, a number $\hat{\gamma} > \gamma^*$ exists such that $\|w_{\hat{\gamma}}\| = r$. The disturbance $w_{\hat{\gamma}}$ is a bounded worst-case disturbance and the supremum (6.2) *equals $x_0^T X_{\hat{\gamma}} x_0 + \hat{\gamma}^2 r^2$.*

Remark 6.2.14 If $\|w_\gamma\|$ tends to infinity for $\gamma \downarrow \gamma^*$, Theorem 6.2.13 gives a solution of Problem 6.2.1 for all $r > 0$. However, if this norm is bounded for $\gamma \downarrow \gamma^*$ and if r is larger than this bound, the situation is unclear. In this case, solutions can possibly be obtained from disturbances maximizing $J_{\gamma^*}(w, x_0)$; see also Remark 6.2.10. More attention is paid to this situation in the next subsection. □

6.2.2 An Operator Approach

Problem 6.2.1 can be put in a more general framework in the following way.

Problem 6.2.15 *Given two real Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, a bounded nonzero linear operator T from $\mathcal{X}$ to $\mathcal{Y}$, an element $y_0 \in \mathcal{Y}$, and a positive real number r, maximize $\|y_0 + Tx\|$ subject to $\|x\| \leq r$. Each $\overline{x} \in \mathcal{X}$ maximizing this criterion is called a solution of this problem.*

In this subsection we first analyze this more general problem, the results of which are summarized in Theorem 6.2.18. The analysis following this theorem concerns the implications of these results on the specific Problem 6.2.1.

Motivated by the analysis preceding Section 6.2.1, we define for each $\mu \in I\!R$ the functional $\psi_\mu : \mathcal{X} \to I\!R$ by

$$\psi_\mu(x) = \|y_0 + Tx\|^2 - \mu^2\|x\|^2.$$

It is easily seen that the results of Theorem 6.2.6 also hold for Problem 6.2.15. Thus we have that $\overline{x} \in \mathcal{X}$ is a solution of Problem 6.2.15 if and only if

(i) $\|\overline{x}\| = r$;

(ii) $\exists\, \mu \geq \|T\| \; \forall\, x \in \mathcal{X} \quad \psi_\mu(x) \leq \psi_\mu(\overline{x})$.

The following lemma provides a necessary and sufficient condition for each $\mu \geq \|T\|$ under which an element $\overline{x} \in \mathcal{X}$ maximizes ψ_μ.

Lemma 6.2.16 *We have*

$$\forall\, x \in \mathcal{X} \; \psi_{\|T\|}(x) \leq \psi_{\|T\|}(\overline{x}) \quad \Leftrightarrow \quad \left(\|T\|^2 - T^*T\right)\overline{x} = T^*y_0.$$

Furthermore, if $\mu > \|T\|$, then

$$\forall\, x \in \mathcal{X} \; \psi_\mu(x) \leq \psi_\mu(\overline{x}) \quad \Leftrightarrow \quad \overline{x} = \left(\mu^2 - T^*T\right)^{-1} T^*y_0.$$

Proof The functional ψ_μ can be written as

$$\psi_\mu(x) = \left\langle x, (T^*T - \mu^2)x \right\rangle + 2\left\langle x, T^* y_0 \right\rangle .$$

Hence according to Theorem 2.1.1, ψ_μ has a maximum if and only if $-T^*T + \mu^2$ is positive semi-definite and $T^* y_0 \in \mathrm{im}(T^*T - \mu^2)$. If $\mu \geq \|T\|$, then the operator $-T^*T + \mu^2$ is positive semi-definite. Theorem 2.1.1 also shows that $\overline{x}$ maximizes ψ_μ if and only if

$$(\mu^2 - T^*T)\overline{x} = T^* y_0.$$

The first statement of the lemma follows by taking $\mu = \|T\|$. The second statement follows from the observation that $\mu^2 - T^*T$ is invertible if $\mu > \|T\|$. □

We deduce that an element $\overline{x} \in \mathcal{X}$ is a solution of Problem 6.2.15 if and only if $\|\overline{x}\| = r$ and

$$\left(\|T\|^2 - T^*T\right)\overline{x} = T^* y_0 \tag{6.20}$$

or

$$\overline{x} = x_\mu := \left(\mu^2 - T^*T\right)^{-1} T^* y_0 \text{ for some } \mu > \|T\|. \tag{6.21}$$

The results formulated in the next lemma are useful for studying whether or not an element $\overline{x} \in \mathcal{X}$ with $\|\overline{x}\| = r$ exists that satisfies either (6.20) or (6.21).

Lemma 6.2.17 *The following properties hold:*

(i) $\|T\|^2$ *is either an eigenvalue of* T^*T*, or it belongs to the continuous spectrum*[1] *of* T^*T.

(ii) If $T^* y_0 \neq 0$*, the norm* $\|x_\mu\|$ *strictly decreases to zero as* μ *increases from* $\|T\|$ *to infinity.*

(iii) If $\|x_\mu\|$ *is unbounded for* $\mu \downarrow \|T\|$*, then* $T^* y_0 \notin \mathrm{im}\,(\|T\|^2 - T^*T)$.

Proof Clearly, T^*T is a bounded self-adjoint linear operator with norm $\|T\|^2$. According to [72, Theorems 9.2-1 and 9.2-3], the maximal value of the spectrum of such an operator equals $\|T\|^2$. Since the residual spectrum of such an operator is empty [72, Theorem 9.2-4], $\|T\|^2$ must either be an eigenvalue of T^*T or it belongs to the continuous spectrum of T^*T.

[1] The continuous spectrum of a linear operator $S : \mathcal{D} \to \mathcal{Z}$, with domain $\mathcal{D}$ a subset of the complex nonzero normed space $\mathcal{Z}$, is defined as the set $\sigma_c(S) \subset \mathbb{C}$ such that for each $\lambda \in \sigma_c(S)$, the operator $(S - \lambda)^{-1}$ exists, is defined on a set which is dense in $\mathcal{Z}$, and is unbounded. See for instance [72, Definition 7.2-1].

To prove the second statement, let $\mu > \|T\|$. Since $\|\mu^{-2}T^*T\| = \mu^{-2}\|T^*T\| = (\|T\|/\mu)^2 < 1$, the inverse of $\mu^2 - T^*T$ can be written as [72, Theorem 7.3-1]

$$\left(\mu^2 - T^*T\right)^{-1} = \mu^{-2}\left(I - \mu^{-2}T^*T\right)^{-1} = \mu^{-2}\sum_{j=0}^{\infty}\left(\mu^{-2}T^*T\right)^j. \tag{6.22}$$

Hence

$$\begin{aligned}
\|x_\mu\|^2 &= \left\langle \left(\mu^2 - T^*T\right)^{-1} T^*y_0, \left(\mu^2 - T^*T\right)^{-1} T^*y_0 \right\rangle = \\
&= \mu^{-4}\left\langle \sum_{j=0}^{\infty}\left(\mu^{-2}T^*T\right)^j T^*y_0, \sum_{j=0}^{\infty}\left(\mu^{-2}T^*T\right)^j T^*y_0 \right\rangle = \\
&= \mu^{-4}\langle T^*y_0, T^*y_0\rangle + \mu^{-6}\langle TT^*y_0, TT^*y_0\rangle + \mu^{-8}\langle T^*TT^*y_0, T^*TT^*y_0\rangle + \cdots \\
&\quad + \mu^{-6}\langle TT^*y_0, TT^*y_0\rangle + \mu^{-8}\langle T^*TT^*y_0, T^*TT^*y_0\rangle + \cdots \\
&\quad + \mu^{-8}\langle T^*TT^*y_0, T^*TT^*y_0\rangle + \cdots \\
&\quad \vdots \\
&= \mu^{-4}\|T^*y_0\|^2 + 2\mu^{-6}\|TT^*y_0\|^2 + 3\mu^{-8}\|T^*TT^*y_0\|^2 + \cdots
\end{aligned}$$

Clearly, $\|x_\mu\|^2$ can be expressed as a Laurent series, which is convergent for $\mu > \|T\|$. The coefficients of this Laurent series are positive if $T^*y_0 \neq 0$, implying that $\|x_\mu\|$ is strictly decreasing in μ if $T^*y_0 \neq 0$. It is moreover clear that $\|x_\mu\| \downarrow 0$ for $\mu \to \infty$.
For the third statement, assume that $T^*y_0 \in \mathrm{im}(\|T\|^2 - T^*T)$ and let $\mu > \|T\|$. Then, there exists an element $x_0 \in \mathcal{X}$ such that $(\|T\|^2 - T^*T)x_0 = T^*y_0$. Thus

$$\begin{aligned}
\|x_\mu\| &= \left\|\left(\mu^2 - T^*T\right)^{-1}\left(\|T\|^2 - T^*T\right)x_0\right\| = \left\|x_0 + (\|T\|^2 - \mu^2)\left(\mu^2 - T^*T\right)^{-1}x_0\right\| \leq \\
&\leq \|x_0\| + (\mu^2 - \|T\|^2)\left\|\left(\mu^2 - T^*T\right)^{-1}\right\|\|x_0\|.
\end{aligned}$$

According to (6.22), we have

$$\left\|\left(\mu^2 - T^*T\right)^{-1}\right\| \leq \mu^{-2}\sum_{j=0}^{\infty}(\|T\|/\mu)^{2j} = \left(\mu^2 - \|T\|^2\right)^{-1}.$$

Hence $\|x_\mu\| \leq 2\|x_0\|$, showing that $\|x_\mu\|$ is bounded; in particular in the limit $\mu \downarrow \|T\|$. □

If $T^*y_0 = 0$, then $x_\mu = 0$ for all $\mu > \|T\|$. Thus in this case solutions of Problem 6.2.15 cannot be obtained from (6.21). However, if $T^*y_0 = 0$, then clearly $T^*y_0 \in \mathrm{im}(\|T\|^2 - T^*T)$. Hence, (6.20) may have nonzero solutions. If $\|T\|^2$ is an eigenvalue of T^*T, the space $\ker(\|T\|^2 - T^*T)$ contains a nonzero element, which can be chosen such that its norm is equal to r. If $\|T\|^2$ belongs to the continuous spectrum of T^*T, the only solution of (6.20) is $\overline{x} = 0$. Hence, in this case, Problem 6.2.15 has no solution.

If $T^*y_0 \neq 0$, the existence of a solution of Problem 6.2.15 that can be obtained from (6.21) depends on the behavior of $\|x_\mu\|$ for $\mu \downarrow \|T\|$. If $\|x_\mu\|$ tends to infinity for $\mu \downarrow \|T\|$, it strictly decreases from ∞ to 0 as μ increases from $\|T\|$ to ∞. Hence, in this case, for each $r > 0$ a unique number $\mu > \|T\|$ such that $\|x_\mu\| = r$ exists. However, if $\|x_\mu\|$ is bounded for $\mu \downarrow \|T\|$ a number $\mu > \|T\|$ for which $\|x_\mu\| = r$ can be found if and only if r is smaller than or equal to this bound.

If $T^*y_0 \neq 0$, and if $\|x_\mu\|$ is bounded for $\mu \downarrow \|T\|$, and $r > \lim_{\mu \downarrow \|T\|} \|w_\mu\|$, solutions of Problem 6.2.15 can only be obtained from (6.20). Whether indeed (6.20) provides solutions depends on whether or not r belongs to the set

$$\Omega := \{\|\overline{x}\| \mid \overline{x} \text{ satisfies (6.20)}\}. \tag{6.23}$$

We summarize the results in the following theorem.

Theorem 6.2.18 *Consider Problem 6.2.15. Let T^* be the adjoint operator of T. Let for each $\mu > \|T\|$, the element $x_\mu \in \mathcal{X}$ be defined by* (6.21). *Then:*

(i) Suppose that $T^*y_0 = 0$. If $\|T\|^2$ is an eigenvalue of the operator T^*T, the solutions of Problem 6.2.15 are given by the elements $\overline{x} \in \ker(\|T\|^2 - T^*T)$ with $\|\overline{x}\| = r$. If $\|T\|^2$ is an element of the continuous spectrum of T^*T, Problem 6.2.15 has no solution.

(ii) Suppose that $T^*y_0 \neq 0$ and $r > \lim_{\mu \downarrow \|T\|} \|x_\mu\|$. If $r \in \Omega$, where Ω is defined by (6.23)*, the solutions of Problem 6.2.15 are given by the solutions $\overline{x}$ of* (6.20) *with $\|\overline{x}\| = r$. If $r \notin \Omega$, Problem 6.2.15 has no solution.*

(iii) Suppose that $T^*y_0 \neq 0$ and $r \leq \lim_{\mu \downarrow \|T\|} \|x_\mu\|$. Then there exists a unique number $\hat{\mu} \geq \|T\|$ such that $\|x_{\hat{\mu}}\| = r$; $x_{\hat{\mu}}$ is a solution of Problem 6.2.15. If $r \notin \Omega$, this is the unique solution of this problem.

The rest of this section deals with the implications of this theorem on Problem 6.2.1, which fits in the formulation of Problem 6.2.15 by setting $\mathcal{X} = L_2^p(0,\infty)$, $\mathcal{Y} = L_2^q(0,\infty)$, $T = \mathcal{G}$, and $y_0 = z_{x_0}$. The variable μ will further be replaced by the variable γ. Note that $\psi_\gamma(x) = J_\gamma(w, x_0)$ and $\|T\| = \gamma^*$.

Remark 6.2.19 Because of uniqueness, it can immediately be concluded from Theorem 6.2.9 that $x_\gamma = w_\gamma$ for each $\gamma > \gamma^*$ with w_γ as defined in this theorem. Thus we have

$$w_\gamma = (\gamma^2 - \mathcal{G}^*\mathcal{G})^{-1}\mathcal{G}^* z_{x_0}, \quad \gamma > \gamma^*.$$

Nevertheless, the following analysis contains a complete derivation of this result. This is done for two reasons. Firstly, the design of the present chapter is such as to make the Sections 6.2.1 and 6.2.2 readable independently from each other. Secondly, the analysis contains useful elements to decide whether or not bounded worst-case disturbances exist if a number $\hat{\gamma}$ cannot be determined from the equation $\|x_\gamma\| = r$; see for instance Example 6.2.24. □

We first determine an expression for $\mathcal{G}^* z_{x_0}$. The adjoint operator $\mathcal{G}^*$ can be determined as follows. Let $w \in L_2^q(0,\infty)$ and $z \in L_2^p(0,\infty)$, then

$$\langle z, \mathcal{G}w\rangle = \int_0^\infty z(t)^T(\mathcal{G}w)(t)dt = \int_0^\infty\int_0^t z(t)^T Ce^{(t-\tau)A}Ew(\tau)d\tau dt =$$
$$= \int_0^\infty\int_\tau^\infty z(t)^T Ce^{(t-\tau)A}Ew(\tau)dtd\tau = \int_0^\infty w(\tau)^T E^T \int_\tau^\infty e^{(t-\tau)A^T}C^T z(t)dtd\tau.$$

In the third step of this computation the order of integration has been interchanged, which is allowed because A is stable. We obtain

$$(\mathcal{G}^* z)(t) = E^T\int_t^\infty e^{(\tau-t)A^T}C^T z(\tau)d\tau. \tag{6.24}$$

Consequently, using (6.3),

$$(\mathcal{G}^* z_{x_0})(t) = E^T e^{-tA^T}\int_t^\infty e^{\tau A^T}C^T Ce^{\tau A}dt x_0 = Ee^{-tA^T}H(t)x_0,$$

where we introduced the notation

$$H(t) := \int_t^\infty e^{\tau A^T}C^T Ce^{\tau A}d\tau = \int_0^\infty e^{(s+t)A^T}C^T Ce^{(s+t)A}ds = e^{tA^T}Pe^{tA}. \tag{6.25}$$

Here P is the unique solution of the Lyapunov equation

$$A^T P + PA = -C^T C. \tag{6.26}$$

Hence,

$$(\mathcal{G}^* z_{x_0})(t) = E^T Pe^{tA}x_0. \tag{6.27}$$

We proceed by analyzing the operator $\gamma^2 - \mathcal{G}^*\mathcal{G}$ in more detail. It will turn out that the operator $\mathcal{I} - \gamma^{-2}\mathcal{G}^*\mathcal{G}$ is a Wiener-Hopf integral operator. Such an operator is defined as follows.

Definition 6.2.20 [54, Section XII.2] A linear operator $\mathcal{T} : L_2^q(0,\infty) \to L_2^q(0,\infty)$ is called a *Wiener-Hopf operator* if $\mathcal{T} = \mathcal{I} - \mathcal{K}$, where $\mathcal{K}$ is an integral operator given by

$$(\mathcal{K}\varphi)(t) = \int_0^\infty k(t-s)\varphi(s)ds, \quad \varphi \in L_2^q(0,\infty). \tag{6.28}$$

The kernel k is a $q \times q$ matrix function which is integrable on the real line. Furthermore, the $q \times q$ matrix function W defined by

$$W(s) = I - \int_{-\infty}^{\infty} e^{ist}k(t)dt, \tag{6.29}$$

is called the *symbol* of the Wiener-Hopf operator $\mathcal{T}$. □

Define

$$\mathcal{K}_\gamma := \gamma^{-2}\mathcal{G}^*\mathcal{G} \tag{6.30}$$

for $\gamma > \gamma^*$. We have

$$\begin{aligned}(\mathcal{K}_\gamma w)(t) &= \gamma^{-2}E^T\int_t^\infty e^{(\tau-t)A}C^T(\mathcal{G}w)(\tau)d\tau = \\ &= \gamma^{-2}E^T\int_t^\infty\int_0^\tau e^{(\tau-t)A^T}C^TCe^{(\tau-s)A}Ew(s)dsd\tau = \\ &= \gamma^{-2}E^T\left(\int_0^t\int_t^\infty + \int_t^\infty\int_s^\infty\right)e^{(\tau-t)A^T}C^TCe^{(\tau-s)A}Ew(s)d\tau ds = \\ &= \gamma^{-2}E^T\left(e^{-tA^T}H(t)\int_0^t e^{-sA}Ew(s)ds + e^{-tA^T}\int_t^\infty H(s)e^{-sA}Ew(s)ds\right).\end{aligned}$$

Recall that H is defined by (6.25), which shows that

$$(\mathcal{K}_\gamma w)(t) = \gamma^{-2}E^T\left(P\int_0^t e^{(t-s)A}Ew(s)ds + \int_t^\infty e^{(s-t)A^T}PEw(s)ds\right). \tag{6.31}$$

Now, define the kernel $k_\gamma : I\!R \to I\!R^{q\times q}$ for $\gamma > \gamma^*$ by

$$k_\gamma(t) = \begin{cases}\gamma^{-2}E^Te^{-tA^T}PE, & t<0;\\ \\ \gamma^{-2}E^TPe^{tA}E, & t>0.\end{cases} \tag{6.32}$$

Then we have

$$(\mathcal{K}_\gamma w)(t) = \int_0^\infty k_\gamma(t-s)w(s)ds.$$

Due to the stability of A, the entries of k are integrable on the real line. Hence, according to Definition 6.2.20, this shows that $\mathcal{I} - \mathcal{K}_\gamma$ is a Wiener-Hopf operator. Denote the symbol of this Wiener-Hopf operator by W_γ. From (6.29) we obtain

$$\begin{aligned} W_\gamma(s) &= I - \int_{-\infty}^{\infty} e^{ist}k_\gamma(t)dt = I - \gamma^{-2}E^T\left(\int_{-\infty}^{0} e^{t(is-A^T)}dtP + P\int_0^\infty e^{t(is+A)}dt\right)E = \\ &= I - \gamma^{-2}E^T\left((is-A^T)^{-1}P - P(is+A)^{-1}\right)E. \end{aligned} \tag{6.33}$$

This symbol is a rational matrix function, i.e. each of the entries of W_γ is a quotient of two polynomials. This rationality enables us to use the following theorem.

Theorem 6.2.21 [54, Chapter XIII, Theorem 7.1] *Let $\mathcal{T}$ be a Wiener-Hopf operator on $L_2^q(0,\infty)$ with a rational symbol W given in realized form, i.e.*

$$W(s) = I + N(s-L)^{-1}M, \quad s \in \mathbb{R} \tag{6.34}$$

for some matrices L, M, and N. Let ν be the size of L, and define $L^\times := L - MN$. Then $\mathcal{T}$ is invertible if and only if $L^\times$ has no real eigenvalues and

$$\mathbb{C}^\nu = \operatorname{im}\Phi \oplus \ker \Phi^\times, \tag{6.35}$$

where Φ and $\Phi^\times$ are the Riesz projections of L and $L^\times$, respectively, corresponding to the eigenvalues in the upper half plane. If these conditions hold, then

$$(\mathcal{T}^{-1}\psi)(t) = \psi(t) + \int_0^\infty \kappa(t,s)\psi(s)ds, \quad t \geq 0, \tag{6.36}$$

with

$$\kappa(t,s) = \begin{cases} iNe^{-itL^\times}\Pi e^{isL^\times}M, & 0 \leq s < t < \infty; \\ \\ -iNe^{-itL^\times}(\mathcal{I}-\Pi)e^{isL^\times}M, & 0 \leq t < s < \infty. \end{cases} \tag{6.37}$$

Here Π is the projection of $\mathbb{C}^\nu$ along $\operatorname{im}\Phi$ onto $\ker\Phi^\times$.

We apply this theorem to the Wiener-Hopf operator $\mathcal{T} = \mathcal{I} - \mathcal{K}_\gamma$ and in particular we use (6.36) with $\psi = \mathcal{G}^* z_{x_0}$ in order to determine

$$x_\gamma = (\gamma^2 - \mathcal{G}^*\mathcal{G})^{-1}\mathcal{G}^* z_{x_0}, \quad \gamma > \gamma^*.$$

Since $\gamma > \gamma^*$ the invertibility of $\mathcal{I} - \mathcal{K}_\gamma$ is ensured, which implies that the two conditions for invertibility, as described in the theorem, do not need to be verified. In order to use expression (6.36), a realized form of W_γ has to be determined. It is easily seen from (6.33) that W_γ can be written as

$$W_\gamma(s) = I + i\gamma^{-2}E^T \begin{bmatrix} P & I \end{bmatrix} \begin{bmatrix} s - iA & 0 \\ 0 & s + iA^T \end{bmatrix}^{-1} \begin{bmatrix} -I \\ P \end{bmatrix} E.$$

Hence, a realization of W_γ in the form (6.34) is given by

$$L = \begin{bmatrix} iA & 0 \\ 0 & -iA^T \end{bmatrix}, \quad N = i\gamma^{-2}E^T \begin{bmatrix} P & I \end{bmatrix}, \quad M = \begin{bmatrix} -I \\ P \end{bmatrix} E. \tag{6.38}$$

Let $L^\times$, Φ, and Π be defined as in Theorem 6.2.21, with L, M, and N given by (6.38). From (6.36), (6.37), and (6.27) we obtain

$$(\mathcal{T}^{-1}\psi)(t) = E^T P e^{tA} x_0 + iNe^{-itL^\times} \left(\int_0^t \Pi - \int_t^\infty (\mathcal{I} - \Pi) \right) e^{isL^\times} M E^T P e^{sA} ds x_0.$$

Note that

$$\frac{d}{ds}\left(e^{isL^\times} \begin{bmatrix} I \\ 0 \end{bmatrix} e^{sA} \right) = e^{isL^\times} \left(iL^\times \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} A \right) e^{sA} = \gamma^{-2} e^{isL^\times} M E^T P e^{sA},$$

yielding

$$\begin{aligned}
(\mathcal{T}^{-1}\psi)(t) =& E^T P e^{tA} x_0 + \\
& + i\gamma^2 N e^{-itL^\times} \left(\Pi e^{isL^\times} \begin{bmatrix} I \\ 0 \end{bmatrix} e^{sA} \Big|_{s=0}^{t} - (\mathcal{I} - \Pi) e^{isL^\times} \begin{bmatrix} I \\ 0 \end{bmatrix} e^{sA} \Big|_{s=t}^{\infty} \right) x_0 = \\
=& E^T P e^{tA} x_0 + \\
& + i\gamma^2 N e^{-itL^\times} \left(e^{itL^\times} \begin{bmatrix} I \\ 0 \end{bmatrix} e^{tA} - \Pi \begin{bmatrix} I \\ 0 \end{bmatrix} - \lim_{s\to\infty} (\mathcal{I} - \Pi) e^{isL^\times} \begin{bmatrix} I \\ 0 \end{bmatrix} e^{sA} \right) x_0 = \\
=& - i\gamma^2 N e^{-itL^\times} \left(\Pi \begin{bmatrix} I \\ 0 \end{bmatrix} + \lim_{s\to\infty} (\mathcal{I} - \Pi) e^{isL^\times} \begin{bmatrix} I \\ 0 \end{bmatrix} e^{sA} \right) x_0.
\end{aligned} \tag{6.39}$$

The next aim is to investigate $e^{-itL^\times}\Pi[I\ \ 0]^T$ and $(\mathcal{I}-\Pi)e^{isL^\times}$ in more detail. Recall from Theorem 6.2.21 that Π is the projection of $\mathbb{C}^{2n}$ along $\operatorname{im}\Phi$ onto $\ker\Phi^\times$, where Φ and $\Phi^\times$ are the Riesz projections of L and $L^\times = L - MN$ respectively, corresponding to the eigenvalues in the upper half plane. Here, L, M and N are defined by (6.38). Due to the block-diagonal structure of L and the stability of A, it is immediately clear that

$$\operatorname{im}\Phi = \operatorname{im}\begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{6.40}$$

Consequently, since $\mathbb{C}^{2n} = \operatorname{im}\Phi \oplus \ker\Phi^\times$, there exists an $n \times n$ matrix Z such that

$$\ker\Phi^\times = \operatorname{im}\begin{bmatrix} I \\ Z \end{bmatrix}. \tag{6.41}$$

We find

$$\Pi\begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ Z \end{bmatrix}. \tag{6.42}$$

Since $\ker\Phi^\times$ is an $L^\times$-invariant subspace of $\mathbb{C}^{2n}$, the matrix representation of $L^\times$ with respect to the basis of $\mathbb{C}^{2n}$ formed by the columns of

$$\begin{bmatrix} I & 0 \\ Z & I \end{bmatrix}$$

has an upper triangular structure. This representation produces the same structure for $e^{-itL^\times}$. For that reason we now determine this matrix representation. Due to the $L^\times$-invariance of $\ker\Phi^\times$, there exists an $n \times n$ matrix Y_{11} such that (using (6.41))

$$L^\times\begin{bmatrix} I \\ Z \end{bmatrix} = \begin{bmatrix} I \\ Z \end{bmatrix} Y_{11}. \tag{6.43}$$

Next, define the $n \times n$ matrices Y_{12} and Y_{22} by the relation

$$L^\times\begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix}\begin{bmatrix} Y_{12} \\ Y_{22} \end{bmatrix}. \tag{6.44}$$

Then we have

$$L^\times\begin{bmatrix} I & 0 \\ Z & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix}\begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix},$$

yielding the desired matrix representation of $L^\times$, i.e.

$$L^\times = \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix} \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Z & I \end{bmatrix}. \tag{6.45}$$

This representation and (6.42) yields

$$\begin{aligned} e^{-itL^\times}\Pi \begin{bmatrix} I \\ 0 \end{bmatrix} &= \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix} \begin{bmatrix} e^{-itY_{11}} & * \\ 0 & e^{-itY_{22}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Z & I \end{bmatrix} \begin{bmatrix} I \\ Z \end{bmatrix} = \\ &= \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix} \begin{bmatrix} e^{-itY_{11}} & * \\ 0 & e^{-itY_{22}} \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix} \begin{bmatrix} e^{-itY_{11}} \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-itY_{11}} \\ Ze^{-itY_{11}} \end{bmatrix}. \end{aligned} \tag{6.46}$$

This result can be further analyzed as follows. We determine Y_{11} explicitly and determine an equation which Z satisfies. For that purpose, we first compute the matrix $L^\times$, which immediately results in

$$L^\times = i \begin{bmatrix} A + \gamma^{-2}EE^TP & \gamma^{-2}EE^T \\ -\gamma^{-2}PEE^TP & -A^T - \gamma^{-2}PEE^T \end{bmatrix}. \tag{6.47}$$

Hence, from (6.43) we obtain

$$Y_{11} = i(A + \gamma^{-2}EE^T(P+Z)) \tag{6.48}$$

and the fact that Z satisfies the equation

$$A^TZ + ZA + \gamma^{-2}(P+Z)EE^T(P+Z) = 0. \tag{6.49}$$

We proceed by investigating $(\mathcal{I} - \Pi)e^{isL^\times}$ in more detail. Because Π is the projection of $\mathbb{C}^{2n}$ along $\operatorname{im}\Phi$ onto $\ker\Phi^\times$, the operator $\mathcal{I} - \Pi$ is the projection of $\mathbb{C}^{2n}$ along $\ker\Phi^\times$ onto $\operatorname{im}\Phi$. From this fact, (6.40), (6.41), and (6.45), we obtain

$$\begin{aligned} (\mathcal{I} - \Pi)e^{isL^\times} &= (\mathcal{I} - \Pi) \begin{bmatrix} I & 0 \\ Z & I \end{bmatrix} \begin{bmatrix} e^{isY_{11}} & * \\ 0 & e^{isY_{22}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Z & I \end{bmatrix} = \\ &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} e^{isY_{11}} & * \\ 0 & e^{isY_{22}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Z & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -e^{isY_{22}}Z & e^{isY_{22}} \end{bmatrix}. \end{aligned} \tag{6.50}$$

Recall from Theorem 6.2.21 that $L^\times$ has no real eigenvalues and that $\operatorname{im}\Phi^\times$ is the spectral subspace corresponding to the eigenvalues of $L^\times$ in the upper half plane. Thus $\ker\Phi^\times$ is the spectral subspace corresponding to the eigenvalues of $L^\times$ in the bottom half plane. Consequently, (6.41) and (6.43) imply that all the eigenvalues of Y_{11} are located in the bottom half plane. Hence, from

(6.45) we obtain that all the eigenvalues of Y_{22} are located in the upper half plane. From this we conclude that all the eigenvalues of iY_{22} are stable. Now, recall from (6.39) that we are interested in the behavior of $(\mathcal{I} - \Pi)e^{isL^\times}$ for $s \to \infty$. Due to the fact that A and iY_{22} are stable and from (6.50) we obtain

$$\lim_{s\to\infty}(\mathcal{I} - \Pi)e^{isL^\times}\begin{bmatrix} I \\ 0 \end{bmatrix} e^{sA} = \lim_{s\to\infty}\begin{bmatrix} 0 & 0 \\ -e^{isY_{22}}Z & e^{isY_{22}} \end{bmatrix}\begin{bmatrix} I \\ 0 \end{bmatrix} e^{sA} = 0.$$

Using this result, (6.46), (6.38), and (6.48) in this order, we obtain from (6.39)

$$\begin{aligned}(\mathcal{T}^{-1}\psi)(t) &= -i\gamma^2 N e^{-itL^\times}\Pi\begin{bmatrix} I \\ 0 \end{bmatrix} x_0 = -i\gamma^2 N\begin{bmatrix} e^{-itY_{11}} \\ Ze^{-itL^\times} \end{bmatrix} x_0 = \\ &= E^T\begin{bmatrix} P & I \end{bmatrix}\begin{bmatrix} e^{-itY_{11}} \\ Ze^{-itL^\times} \end{bmatrix} x_0 = E^T(P+Z)e^{t(A+\gamma^{-2}EE^T(P+Z))}x_0. \end{aligned} \tag{6.51}$$

Next, by adding (6.26) and (6.49) we see that $X := P + Z$ satisfies the algebraic Riccati equation

$$C^TC + A^TX + XA + \gamma^{-2}XEE^TX = 0. \tag{6.52}$$

Furthermore, recall that Y_{11} has all its eigenvalues in the bottom half plane. Consequently, the matrix $A + \gamma^{-2}EE^TX = -iY_{11}$ (see (6.48)) is a stable matrix. This means that X is a stabilizing solution of (6.52), which is a unique solution. We arrive at the following result.

Theorem 6.2.22 *Consider the linear system* (6.1) *with A stable and denote its H_∞ norm by γ^*. Consider Problem 6.2.1. Define for each $\gamma > \gamma^*$, $w_\gamma \in L_2^q(0,\infty)$ by*

$$w_\gamma(t) = \gamma^{-2}E^TX_\gamma e^{t(A+\gamma^{-2}EE^TX_\gamma)}x_0,$$

where X_γ is the stabilizing solution of the algebraic Riccati equation

$$C^TC + A^TX + XA + \gamma^{-2}XEE^TX = 0.$$

Let P be the solution of the Lyapunov equation $A^TP + PA = -C^TC$, and let Q be the solution of the Lyapunov equation $A^TQ + QA = -PEE^TP$. Assume that $x_0^TQx_0 \neq 0$. For each positive number $r < \lim_{\gamma\downarrow\gamma^*}\|w_\gamma\|$, a unique number $\hat{\gamma} > \gamma^*$ exists such that $\|w_{\hat{\gamma}}\| = r$. In this case, $w_{\hat{\gamma}}$ is a bounded worst-case disturbance and the supremum (6.2) *is equal to $x_0^TX_{\hat{\gamma}}x_0 + \hat{\gamma}^2r^2$. If $\|w_\gamma\|$ is not bounded for $\gamma \downarrow \gamma^*$, then $w_{\hat{\gamma}}$ is a unique bounded worst-case disturbance.*

Proof This theorem is based on Theorem 6.28. According to the analysis above we have $w_\gamma = x_\gamma = \mathcal{T}^{-1}\psi$. From the expression (6.27) it is easily seen that the condition $\mathcal{G}^* z_{x_0} \neq 0$ is equivalent to the condition $x_0^T Q x_0 \neq 0$. The rest immediately follows from Theorem 6.28. $\square$

Although this theorem gives more precise existence and uniqueness statements than Theorem 6.2.13, it does not answer the question whether or not a bounded worst-case disturbance exists if $r > \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$. For small dimensions of the system data, one can study this question analytically. After the next theorem we present an example, which uses the result of the next theorem, showing that bounded worst-case disturbances may not exist if $r > \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$.

Theorem 6.2.23 *The solution of the integral equation*

$$w(t) = y(t) + \mu \int_0^\infty k(t-\tau) w(\tau) d\tau, \quad t > 0, \tag{6.53}$$

with $y(t) = \eta e^{-at}$, $k(s) = e^{-a|s|}$, *and* $\mu, \eta, a \in I\!R$, *with* $a, \mu > 0$, *is given by*

$$w(t) = \begin{cases} \dfrac{2a\eta}{a+\rho} e^{-\rho t}, & a > 2\mu, \\ -2a\eta t + A(1+at), & a = 2\mu, \\ -\dfrac{2a\eta}{\rho} \sin \rho t + A(\rho \cos \rho t + a \sin \rho t), & a < 2\mu, \end{cases}$$

with $\rho = \sqrt{|a^2 - 2a\mu|}$, *and* A *an arbitrary constant.*

Proof This integral equation can be solved using complex Fourier transforms and the Wiener-Hopf technique [95]. Assume first that $a > 2\mu$ and define $\rho = \sqrt{a^2 - 2a\mu}$. We introduce the following notations:

$$y_+(t) = \begin{cases} y(t), & t > 0, \\ 0, & t < 0; \end{cases}$$

$$w_+(t) = \begin{cases} w(t), & t > 0, \\ 0, & t < 0; \end{cases} \qquad w_-(t) = \begin{cases} 0, & t > 0, \\ \mu \int_0^\infty k(t-\tau) w(\tau) d\tau, & t < 0. \end{cases}$$

The integral equation (6.53) can now be written as

$$w_+(t) + w_-(t) = y_+(t) + \mu \int_{-\infty}^{\infty} k(t-\tau) w_+(\tau) d\tau, \quad -\infty < t < \infty. \tag{6.54}$$

We apply a complex Fourier transformation to this equation. The complex variable in the Fourier domain will be denoted by ζ. Let K, Y_+, W_+ and W_- denote the Fourier transforms of k, y_+, w_+ and w_-, respectively, i.e.

$$K(\zeta) = \int_{-\infty}^{\infty} k(s)e^{i\zeta s}ds = \frac{2a}{\zeta^2 + a^2},$$

$$Y_+(\zeta) = \int_{0}^{\infty} y_+(t)e^{i\zeta t}dt = \frac{i\eta}{\zeta + ia},$$

$$W_+(\zeta) = \int_{0}^{\infty} w_+(t)e^{i\zeta t}dt,$$

$$W_-(\zeta) = \int_{-\infty}^{0} w_-(t)e^{i\zeta t}dt.$$

Note that $K(\zeta)$ is analytic for $-a < \operatorname{Im}\zeta < a$, and that $Y_+(\zeta)$ is analytic for $\operatorname{Im}\zeta > -a$. From the theory of Fourier transforms it follows moreover that $W_+(\zeta)$ is analytic for $\operatorname{Im}\zeta > \alpha_1$, and that $W_-(\zeta)$ is analytic for $\operatorname{Im}\zeta < \beta_1$, for some numbers α_1 and β_1. These numbers will be determined later. Let $\alpha = \max\{-a, \alpha_1\}$ and $\beta = \min\{a, \beta_1\}$. Assume that $\alpha < \beta$. We verify this condition after the numbers α_1 and β_1 have been determined. Applying complex Fourier transformation to equation (6.54) yields

$$W_+(\zeta) + W_-(\zeta) = Y_+(\zeta) + \mu K(\zeta)W_+(\zeta), \quad \alpha < \operatorname{Im}\zeta < \beta. \tag{6.55}$$

Note that all the appearing functions are analytic in the strip $\alpha < \operatorname{Im}\zeta < \beta$. This is a required condition for the Wiener-Hopf technique. A second condition for this technique is $1 - \mu K(\zeta) \neq 0$ in the same strip or in a 'sub'-strip. We have

$$1 - \mu K(\zeta) = \frac{\zeta^2 + \rho^2}{\zeta^2 + a^2}. \tag{6.56}$$

Apparently, the second condition for the Wiener-Hopf technique is satisfied if we redefine α and β by $\alpha = \max\{-\rho, \alpha_1\}$ and $\beta = \min\{\rho, \beta_1\}$ (note that $\rho < a$). We still have to check the condition $\alpha < \beta$ after the determination of α_1 and β_1. The next step in the Wiener-Hopf technique is a factorization $1 - \mu K(\zeta) = K_+(\zeta)/K_-(\zeta)$, with $K_+(\zeta)$ analytic and $\neq 0$ for $\operatorname{Im}\zeta > \alpha$ and $K_-(\zeta)$ analytic and $\neq 0$ for $\operatorname{Im}\zeta < \beta$. By inspection of (6.56) we see that such a factorization is given by

$$K_+(\zeta) = \frac{\zeta + i\rho}{\zeta + ia}, \quad K_-(\zeta) = \frac{\zeta - ia}{\zeta - i\rho}.$$

Multiplying (6.55) by $K_-(\zeta)$ yields the equation

$$K_+(\zeta)W_+(\zeta) + K_-(\zeta)W_-(\zeta) - K_-(\zeta)Y_+(\zeta) = 0, \quad \alpha < \operatorname{Im}\zeta < \beta. \tag{6.57}$$

The next step is a decomposition $-K_-(\zeta)Y_+(\zeta) = D_+(\zeta) + D_-(\zeta)$, with $D_+(\zeta)$ analytic for $\operatorname{Im}\zeta > \alpha$ and $D_-(\zeta)$ analytic for $\operatorname{Im}\zeta < \beta$. We have

$$-K_-(\zeta)Y_+(\zeta) = \frac{-i\eta\zeta - a\eta}{(\zeta - i\rho)(\zeta + ia)} = \frac{-2ia\eta/(a+\rho)}{\zeta + ia} + \frac{i\eta(a-\rho)/(a+\rho)}{\zeta - i\rho}.$$

Hence, a required decomposition is given by

$$D_+(\zeta) = \frac{-2ia\eta/(a+\rho)}{\zeta + ia}, \quad D_-(\zeta) = \frac{i\eta(a-\rho)/(a+\rho)}{\zeta - i\rho}.$$

Now, rewrite (6.57) as

$$K_+(\zeta)W_+(\zeta) + D_+(\zeta) = -K_-(\zeta)W_-(\zeta) - D_-(\zeta), \quad \alpha < \operatorname{Im}\zeta < \beta. \tag{6.58}$$

Note that the left-hand side is analytic for $\operatorname{Im}\zeta > \alpha$ and that the right-hand side is analytic for $\operatorname{Im}\zeta < \beta$. Moreover, left-hand side and right-hand side are equal in the strip intersection $\alpha < \operatorname{Im}\zeta < \beta$. Hence, left-hand side and right-hand side are each other's analytic continuation. This implies that the function $J(\zeta)$, defined by

$$J(\zeta) = \begin{cases} K_+(\zeta)W_+(\zeta) + D_+(\zeta), & \operatorname{Im}\zeta > \alpha, \\ -K_-(\zeta)W_-(\zeta) - D_-(\zeta), & \operatorname{Im}\zeta < \beta, \end{cases} \tag{6.59}$$

is an entire function. From the lemma of Riemann-Lebesgue we know that $W_+(\zeta) = o(1)$ for $|\zeta| \to \infty$, $\operatorname{Im}\zeta > \alpha$ (see for instance [29] for an explanation of the o and O symbol). Since $K_+(\zeta) = O(1)$ and $D_+(\zeta) = O(\zeta^{-1})$ for $|\zeta| \to \infty$, $\operatorname{Im}\zeta > \alpha$, we have $J(\zeta) = O(\zeta^{-\delta_1})$, for $|\zeta| \to \infty$, $\operatorname{Im}\zeta > \alpha$, and for some $\delta_1 > 0$. Similar reasoning gives $J(\zeta) = O(\zeta^{-\delta_2})$, for $|\zeta| \to \infty$, $\operatorname{Im}\zeta < \beta$, and for some $\delta_2 > 0$. Hence, from Liouville's theorem it follows that $J(\zeta) = 0$ for all $\zeta \in \mathbb{C}$. This implies that

$$W_+(\zeta) = -\frac{D_+(\zeta)}{K_+(\zeta)} = \frac{2ia\eta/(a+\rho)}{\zeta + i\rho}, \quad \operatorname{Im}\zeta > \alpha. \tag{6.60}$$

Note that $\alpha_1 = -\rho$, so that $\alpha = -\rho$. One can also determine $W_-(\zeta)$ from (6.59). From the resulting expression it is then easily seen that $\beta_1 = a$, so that $\beta = \rho$. We see that since $\rho > 0$, the condition $\alpha < \beta$ is indeed satisfied. The final step consists of determining w by taking the inverse Fourier transform of $W_+(\zeta)$. From (6.60), we have for $t > 0$, and for $\sigma > -\rho$,

$$w(t) = \frac{1}{2\pi} \int_{-\infty+i\sigma}^{\infty+i\sigma} W_+(\zeta)e^{-i\zeta t}d\zeta = \frac{ia\eta}{\pi(a+\rho)} \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{e^{-i\zeta t}}{\zeta + i\rho}d\zeta.$$

This integral can easily be computed by closing the integration path by a semicircle with radius R in the bottom half plane. The contribution of the semicircle converges to zero as $R \to \infty$, which follows from the lemma of Jordan. From the residue theorem of Cauchy it then follows that

$$w(t) = \frac{2a\eta}{a+\rho} \operatorname{Res}_{\zeta=-i\rho} \frac{e^{-i\zeta t}}{\zeta + i\rho} = \frac{2a\eta}{a+\rho} e^{-\rho t}.$$

Next, assume that $a \leq 2\mu$. Define $\rho = \sqrt{2\mu a - a^2}$. Structurally, the derivation of the solution consists of the same steps as in the previous case. A first difference appears in the expression for $1 - \mu K(\zeta)$. Indeed, in the present case we have

$$1 - \mu K(\zeta) = \frac{\zeta^2 - \rho^2}{\zeta^2 + a^2}.$$

For the Wiener-Hopf technique to be successful, this function needs to be analytic and $\neq 0$ in a 'sub'-strip of $\alpha < \operatorname{Im} \zeta < \beta$. Note that $1 - \mu K(\zeta)$ has two zeros ($\zeta = \pm\rho$) on the real axis. Therefore, we redefine α as $\alpha = \max\{0, \alpha_1\}$ and assume that $\alpha < \beta$. Again, this condition needs to be verified later. The factorization now takes the form:

$$K_+(\zeta) = \frac{\zeta^2 - \rho^2}{\zeta + ia}, \quad K_-(\zeta) = w - ia.$$

Note that $1 - \mu K(\zeta) = K_+(\zeta)/K_-(\zeta)$, that $K_+(\zeta)$ is $\neq 0$ and analytic for $\operatorname{Im} \zeta > \alpha$, and that $K_-(\zeta)$ is $\neq 0$ and analytic for $\operatorname{Im} \zeta < \beta$. The next step is the decomposition $-K_-(\zeta)Y_+(\zeta) = D_+(\zeta) + D_-(\zeta)$. It is easily seen that a successful decomposition is given by

$$D_+(\zeta) = -\frac{2a\eta}{\zeta + ia}, \quad D_-(\zeta) = -i\eta.$$

The function $J(\zeta)$ is now defined by

$$J(\zeta) = \begin{cases} \dfrac{\zeta^2 - \rho^2}{\zeta + ia} W_+(\zeta) - \dfrac{2a\eta}{\zeta + ia}, & \operatorname{Im} \zeta > \alpha, \\[2em] -(\zeta - ia) W_-(\zeta) + i\eta, & \operatorname{Im} \zeta < \beta. \end{cases}$$

Again we have the result that $J(\zeta)$ is an entire function. The behavior of $J(\zeta)$ at infinity is

$$J(\zeta) = \begin{cases} o(\zeta) + O(\zeta^{-1}), & |\zeta| \to \infty, \quad \operatorname{Im} \zeta > \alpha, \\ o(\zeta) + O(1), & |\zeta| \to \infty, \quad \operatorname{Im} \zeta < \beta. \end{cases}$$

Hence, Liouville's theorem gives $J(\zeta) = A^*$, with A^* an arbitrary constant. This implies that

$$W_+(\zeta) = \frac{2a\eta}{\zeta^2 - \rho^2} + A^* \frac{\zeta + ia}{\zeta^2 - \rho^2}, \quad W_-(\zeta) = \frac{i\eta - A^*}{\zeta - ia}.$$

Note that $W_+(\zeta)$ is analytic for $\operatorname{Im}\zeta > 0$, so that $\alpha_1 = 0$, and thus $\alpha = 0$. Moreover, $W_-(\zeta)$ is analytic for $\operatorname{Im}\zeta < a$, hence $\beta_1 = a$, and thus $\beta = a$. We conclude that the condition $\alpha < \beta$ is indeed satisfied. By inverse Fourier transformation we find (the integrals are calculated in the same manner as in the previous case) for $t > 0$ and $\sigma > 0$,

$$
\begin{aligned}
w(t) &= \frac{1}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} W_+(\zeta)e^{-it\zeta}d\zeta = \\
&= \frac{a\eta}{\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{e^{-it\zeta}}{\zeta^2-\rho^2}d\zeta + \frac{A^*}{2\pi}\int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{(\zeta+ia)e^{-it\zeta}}{\zeta^2-\rho^2}d\zeta = \\
&= -2a\eta i\left(\operatorname{Res}_{\zeta=-\rho} + \operatorname{Res}_{\zeta=\rho}\right)\frac{e^{-it\zeta}}{\zeta^2-\rho^2} - iA\left(\operatorname{Res}_{\zeta=-\rho} + \operatorname{Res}_{\zeta=\rho}\right)\frac{(\zeta+ia)e^{-it\zeta}}{\zeta^2-\rho^2}.
\end{aligned}
$$

Obviously, we have to distinguish between the cases $\rho > 0$ and $\rho = 0$. Assume first that $\rho > 0$, then the poles are simple and we find

$$
w(t) = -\frac{2a\eta}{\rho}\sin\rho t + A(\rho\cos\rho t + a\sin\rho t),
$$

with $A = -A^* i/\rho$ an arbitrary real constant. If $\rho = 0$, the poles are double and we find

$$
w(t) = -2a\eta t + A(1 + at),
$$

with $A = -A^* i$ an arbitrary real constant. □

The following example shows the existence of a situation in which bounded worst-case disturbances do not exist.

Example 6.2.24 Let $n = q = p = 2$ and

$$
A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad E = C = I, \quad x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
$$

The solution of the Lyapunov equation (6.26) is

$$
P = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}.
$$

Thus from Theorem 6.2.22 we obtain

$$
(\mathcal{G}^* z_{x_0})(t) = \begin{bmatrix} 0 \\ \frac{1}{4}e^{-2t} \end{bmatrix}.
$$

The stabilizing solution of the algebraic Riccati equation (6.52) is

$$X = \begin{bmatrix} \gamma^2\left(1-\sqrt{1-\gamma^{-2}}\right) & 0 \\ 0 & \gamma^2\left(2-\sqrt{4-\gamma^{-2}}\right) \end{bmatrix},$$

which exists if and only if $\gamma > 1$. Hence $\gamma^* = 1$. For $\gamma > \gamma^*$ we have

$$w_\gamma(t) = \begin{bmatrix} 0 \\ \left(2-\sqrt{4-\gamma^{-2}}\right)e^{-t\sqrt{4-\gamma^{-2}}} \end{bmatrix}.$$

Thus

$$\|w_\gamma\|^2 = \frac{8-\gamma^{-2}-4\sqrt{4-\gamma^{-2}}}{2\sqrt{4-\gamma^{-2}}}.$$

So in this example the norm of w_γ is bounded for $\gamma \downarrow \gamma^*$. We have

$$\lim_{\gamma\downarrow\gamma^*} \|w_\gamma\|^2 = -2+\frac{7}{6}\sqrt{3}.$$

Now, let $r > \sqrt{-2+\frac{7}{6}\sqrt{3}}$. Note that in this example maximizing disturbances can only be found according to case (ii) of Theorem 6.2.18, which states that a maximizing disturbance exists if and only if $r \in \Omega$. In order to determine the set Ω, consider equation (6.20). From (6.30) and (6.31) we derive

$$(\mathcal{G}^*\mathcal{G}w)(t) = \begin{bmatrix} \frac{1}{2}\int_0^\infty e^{-|t-s|}w^1(s)ds \\ \frac{1}{4}\int_0^\infty e^{-2|t-s|}w^2(s)ds \end{bmatrix}, \quad w = \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}.$$

Hence, equation (6.20) with $\overline{w} = [\overline{w}^1 \;\; \overline{w}^2]^T$ reduces to the two decoupled integral equations

$$\overline{w}^1(t) - \frac{1}{2}\int_0^\infty e^{-|t-s|}\overline{w}^1(s)ds = 0,$$

$$\overline{w}^2(t) - \frac{1}{4}\int_0^\infty e^{-2|t-s|}\overline{w}^2(s)ds = \frac{1}{4}e^{-2t}.$$

According to Theorem 6.2.23, the only solution of the first integral equation in $L_2(0,\infty)$ is the trivial solution, i.e. $\overline{w}^1 = 0$. The second integral equation has the unique solution

$$\overline{w}^2(t) = (2-\sqrt{3})e^{-t\sqrt{3}}.$$

Since $\|\overline{w}\|^2 = \|\overline{w}^2\|^2 = -2+\frac{7}{6}\sqrt{3}$, we have $\Omega = \left\{\sqrt{-2+\frac{7}{6}\sqrt{3}}\right\}$ (this set is defined by (6.23)). Hence $r \notin \Omega$ and thus, according to Theorem 6.2.18, the supremum in Problem 6.2.1 is not a maximum. Note that the only solution of (6.20) in $L_2^2(0,\infty)$ has norm equal to the number $\lim_{\gamma\downarrow\gamma^*}\|w_\gamma\|$. In fact, we have

$$\lim_{\gamma\downarrow\gamma^*} w_\gamma(t) = \begin{bmatrix} 0 \\ (2-\sqrt{3})e^{-t\sqrt{3}} \end{bmatrix} = \overline{w}(t).$$

□

6.3 Worst-Case Disturbance Attenuation with Nonzero Initial State

In this section we study the situation with one player minimizing his cost function which is maximized by the bounded disturbance. He minimizes the L_2-norm of the output of a linear system by choosing an appropriate linear time-invariant feedback matrix which internally stabilizes the system. Having the H_∞ theory in mind, we will study the existence of a saddle-point solution instead of explicitly studying the minmax problem. Under certain conditions a saddle-point solution indeed exists and this solution serves as a starting point to study the N-player situation in the next section. For the notations and terminology used in the present section we refer to Chapter 2.

The linear system is described by the equations

$$\dot{x} = (A+BF)x + Ew, \quad x(0) = x_0, \quad z = (C+DF)x. \tag{6.61}$$

We assume that (A,B) is stabilizable and that D is injective; the latter assumption is usually referred to as the regular case. The control feedback matrix F is assumed to be in the set $\mathcal{F}$. By definition, the stabilizability of (A,B) ensures that $\mathcal{F}\neq\emptyset$. The disturbance w is assumed to be an element of the Hilbert space $L_2^q(0,\infty)$. Consequently, due to the stability of $A+BF$, the output z is an element of the Hilbert space $L_2^p(0,\infty)$. The output for zero initial state depends linearly on the disturbance, i.e. it is the image of the linear operator $\mathcal{G}_F : L_2^q(0,\infty)\rightarrow L_2^p(0,\infty)$, defined for a fixed feedback matrix $F\in\mathcal{F}$ by

$$(\mathcal{G}_F w)(t) = (C+DF)\int_0^t e^{(t-\tau)(A+BF)}Ew(\tau)d\tau. \tag{6.62}$$

Due to the stability of $A+BF$, this operator is a well-defined bounded linear operator; for more details see the text between (6.4) and (6.5). Next, let

$$\gamma^* := \inf_{F \in \mathcal{F}} \|\mathcal{G}_F\|; \tag{6.63}$$

$$J(F, w, x_0) := \|z\|^2; \tag{6.64}$$

$$B_r := \{w \in L_2^q(0, \infty) | \, \|w\| \leq r\}, \quad r > 0. \tag{6.65}$$

Note that the notation γ^* was used in the previous section to denote the norm of the linear operator $\mathcal{G}$ (see (6.4) and (6.5)). In the present situation this operator depends on the feedback matrix F and we use the notation γ^* to denote the minimal achievable norm, which is in line with the conventions in H_∞ control theory. The optimization problem to be studied in the present section can now be formulated as follows.

Problem 6.3.1 Let $r > 0$. Determine, if it exists, a saddle-point solution for the two-person zero-sum game

$$(\mathcal{F}, B_r, J(\cdot, \cdot, x_0)) \,. \tag{6.66}$$

Furthermore, if such a saddle-point solution exists, determine the value of this game. □

See Section 2.1 for the definition of a two-person zero-sum game and a number of related aspects. Problem 6.3.1 is related to the state feedback H_∞ control problem in the following way. If $x_0 = 0$, then

$$\inf_{F \in \mathcal{F}} \sup_{w \in B_r} \|z\|^2 = r^2 \inf_{F \in \mathcal{F}} \|\mathcal{G}_F\|^2. \tag{6.67}$$

The norm of the operator $\mathcal{G}_F$ is known to be equal to the H_∞ norm of the closed-loop system (6.61). Hence, the minimal achievable value of this norm, i.e. r^{-1} times the square root of (6.67), equals the upper value of the two-person zero-sum game (6.66) with $x_0 = 0$. Actually, a saddle-point solution for this game, i.e. for the game (6.66) with $x_0 = 0$, does not exist [9, Section 1.4]. If there exists a saddle-point solution for $x_0 \neq 0$, upper and lower value are equal, and hence, such a solution solves the state feedback H_∞ control problem for nonzero initial state. Actually, the term H_∞ control problem is inappropriate here since the problem does not minimize an H_∞ norm in case of nonzero initial state. A better terminology is: A saddle-point solution for the game (6.66) solves a worst-case disturbance attenuation problem for linear systems with nonzero initial state.

The game (6.66) is related to the corresponding *soft-constrained differential game* [9], which is the two-person zero-sum game

$$(\mathcal{F}, L_2^q(0, \infty), J_\gamma(\cdot, \cdot, x_0)). \tag{6.68}$$

where the criterion $J_\gamma : \mathcal{F} \times L_2^q(0,\infty) \times I\!R^n \to I\!R$ is defined by

$$J_\gamma(F, w, x_0) = J(F, w, x_0) - \gamma^2 \|w\|^2, \quad \gamma \in I\!R. \tag{6.69}$$

The relation between the games (6.66) and (6.68) that is useful for solving Problem 6.3.1 is made precise in the following lemma.

Lemma 6.3.2 *A saddle-point solution* $(\overline{F}, \overline{w})$ *with* $\|\overline{w}\| = r$ *for the game* (6.68) *is also a saddle-point solution for the game* (6.66).

Proof Let $w \in B_r$, then

$$J(w, \overline{F}, x_0) = J_\gamma(w, \overline{F}, x_0) + \gamma^2 \|w\|^2 \le J_\gamma(\overline{w}, \overline{F}, x_0) + \gamma^2 r^2 = J(\overline{w}, \overline{F}, x_0).$$

Next, let $F \in \mathcal{F}$, then

$$J(\overline{w}, \overline{F}, x_0) = J_\gamma(\overline{w}, \overline{F}, x_0) + \gamma^2 \|\overline{w}\|^2 \le J_\gamma(\overline{w}, F, x_0) + \gamma^2 \|\overline{w}\|^2 = J(\overline{w}, F, x_0).$$

Hence, we obtain $J(w, \overline{F}, x_0) \le J(\overline{w}, \overline{F}, x_0) \le J(\overline{w}, F, x_0)$ for all $w \in B_r$ and for all $F \in \mathcal{F}$, which shows that $(\overline{F}, \overline{w})$ is a saddle-point solution for the game (6.66). □

Motivated by this lemma we approach Problem 6.3.1 as follows. Firstly, we look for saddle-point solutions for the soft-constrained game $(\mathcal{F}, L_2^q(0,\infty), J_\gamma(\cdot,\cdot,x_0))$ with γ arbitrary. Secondly, we investigate by varying γ whether saddle-point solutions $(\overline{F}, \overline{w})$ with $\|\overline{w}\| = r$ exist.

Remark 6.3.3 A saddle-point solution for the soft-constrained game with a closed-loop information pattern for both players is known to exist under some weak conditions [9, Chapter 4]. The information structure for the maximizing player here is an open-loop pattern and the minimizing player has a memoryless perfect state pattern. □

A saddle-point solution for the soft-constrained game can be found by two completions of the square. In order to see this, let X be an arbitrary symmetric $n \times n$ matrix. Then

$$
\begin{aligned}
J_\gamma(F, w, x_0) &= \|z\|^2 - \gamma^2 \|w\|^2 = \\
&= \int_0^\infty \left(x^T (C + DF)^T (C + DF) x - \gamma^2 w^T w + \frac{d}{dt} x^T X x - \frac{d}{dt} x^T X x \right) dt \\
&= x_0^T X x_0 + \int_0^\infty \Big(x^T ((C + DF)^T (C + DF) + (A + BF)^T X + \\
&\qquad\qquad + X(A + BF)) x + 2 w^T E^T X x - \gamma^2 w^T w \Big)\, dt = \\
&= x_0^T X x_0 + \int_0^\infty \Big(x^T (C^T C + A^T X + X A + F^T D^T D F + 2 F^T D^T C + \\
&\qquad\qquad + 2 F^T B^T X) x - \gamma^2 w^T w + 2 w^T E^T X x \Big)\, dt. \qquad (6.70)
\end{aligned}
$$

The third step holds because $x(t) \to 0$ for $t \to \infty$; see Lemma 2.1.2. The derived expression shows how the two squares should be completed. Indeed, since D is injective,

$$
\begin{aligned}
& x^T (F^T D^T D F + 2 F^T D^T C + 2 F^T B^T X) x = \\
&= \left| D(F + (D^T D)^{-1} (D^T C + B^T X)) x \right|^2 - x^T (C^T D + X B)(D^T D)^{-1} (D^T C + B^T X) x
\end{aligned}
$$

and, assuming that $\gamma \neq 0$,

$$
-\gamma^2 w^T w + 2 w^T E^T X x = -\gamma^2 \left| w - \gamma^{-2} E^T X x \right|^2 + \gamma^{-2} x^T X E E^T X x.
$$

Substituting these equalities in (6.70) yields

$$
\begin{aligned}
J_\gamma(F, w, x_0) &= x_0^T X x_0 + \int_0^\infty \Big(x^T (Q + \overline{A}^T X + X \overline{A} - X S X + \gamma^{-2} X E E^T X) x + \\
&\quad + \left| D(F + (D^T D)^{-1} (D^T C + B^T X)) x \right|^2 - \gamma^2 \left| w - \gamma^{-2} E^T X x \right|^2 \Big)\, dt, \qquad (6.71)
\end{aligned}
$$

with

$$Q := C^T (I - D(D^T D)^{-1} D^T) C, \qquad (6.72)$$

$$\overline{A} := A - B(D^T D)^{-1} D^T C, \qquad (6.73)$$

$$S := B(D^T D)^{-1} B^T. \qquad (6.74)$$

Hence, if X satisfies the algebraic Riccati equation

$$Q + \overline{A}^T X + X \overline{A} - X S X + \gamma^{-2} X E E^T X = 0, \qquad (6.75)$$

then

$$J_\gamma(F, w, x_0) = x_0^T X x_0 + \int_0^\infty \left| D(F + (D^T D)^{-1}(D^T C + B^T X))x \right|^2 dt + \\ - \gamma^2 \int_0^\infty \left| w - \gamma^{-2} E^T X x \right|^2 dt. \tag{6.76}$$

Obviously, the feedback matrix

$$\overline{F} := -(D^T D)^{-1}(D^T C + B^T X) \tag{6.77}$$

satisfies the maximization property of a saddle-point solution (property (6.82) below). It is however unclear whether this matrix is an element of $\mathcal{F}$. This can be accomplished by further specifying the matrix X according to the next lemma. This lemma can for instance be found in [128, Lemma 16.6].

Lemma 6.3.4 *Let X be a symmetric positive semi-definite solution of the algebraic Riccati equation* (6.75) *and assume that the matrix $\overline{A} - SX + \gamma^{-2} E E^T X$ is stable. Then the matrix $\overline{A} - SX$ is stable.*

The matrix $\overline{A} - SX + \gamma^{-2} E E^T X$ is stable if and only if X is the stabilizing solution of the algebraic Riccati equation (6.75). Let this be the case in the sequel. Then $\overline{F} \in \mathcal{F}$ and (6.76) yields

$$J_\gamma(\overline{F}, w, x_0) = x_0^T X x_0 - \gamma^2 \int_0^\infty \left| w - \gamma^{-2} E^T X \tilde{x} \right|^2 dt \leq x_0^T X x_0, \tag{6.78}$$

for all $w \in L_2^q(0, \infty)$. Here, $\tilde{x}$ denotes the state generated by

$$\dot{\tilde{x}} = (A + B\overline{F})\tilde{x} + Ew, \quad \tilde{x}(0) = x_0. \tag{6.79}$$

The result (6.78) implies that if $J_\gamma(\overline{F}, w, x_0) = x_0^T X x_0$, then $w = \gamma^{-2} E^T X \tilde{x}$. Substituting this in (6.79) yields $w = \overline{w}$, where

$$\overline{w}(t) := \gamma^{-2} E^T X e^{t(A + B\overline{F} + \gamma^{-2} E E^T X)} x_0. \tag{6.80}$$

Equivalently, $\overline{w} = \gamma^{-2} E^T X \overline{x}$, where $\overline{x}$ denotes the state corresponding to $F = \overline{F}$ and $w = \overline{w}$, i.e. $\overline{x}$ is generated by

$$\dot{\overline{x}} = (A + B\overline{F})\overline{x} + E\overline{w}, \quad \overline{x}(0) = x_0. \tag{6.81}$$

Note that $\overline{w} \in L_2^q(0,\infty)$ since $A + B\overline{F} + \gamma^{-2}EE^TX$ is stable. We obtain

$$J_\gamma(\overline{F}, w, x_0) < x_0^T X x_0 \text{ for all } w \neq \overline{w}, \text{ and } J_\gamma(\overline{F}, \overline{w}, x_0) = x_0^T X x_0. \tag{6.82}$$

The next aim is to derive that $\overline{w}$ satisfies the minimization property of a saddle-point solution, i.e. to prove that $J_\gamma(F, \overline{w}, x_0) - J_\gamma(\overline{F}, \overline{w}, x_0)$ is nonnegative for each $F \in \mathcal{F}$. Let $\hat{x}$ denote the state corresponding to $w = \overline{w}$ and an arbitrary feedback matrix $F \in \mathcal{F}$, i.e. $\hat{x}$ is generated by

$$\dot{\hat{x}} = (A + BF)\hat{x} + E\overline{w}, \quad \hat{x}(0) = x_0. \tag{6.83}$$

Define furthermore

$$\nu := D(\overline{F} - F)\hat{x},$$
$$\zeta := \overline{w} - \gamma^{-2}E^TX\hat{x}.$$

Then, (6.76) implies that

$$J_\gamma(F, \overline{w}, x_0) - J_\gamma(\overline{F}, \overline{w}, x_0) = \int_0^\infty |\nu|^2 dt - \gamma^2 \int_0^\infty |\zeta|^2 dt = \|\nu\|^2 - \gamma^2\|\zeta\|^2. \tag{6.84}$$

Yet another completion of the square is required to show that this expression is nonnegative for all feedback matrices $F \in \mathcal{F}$. In preparation for that, let $\xi := \overline{x} - \hat{x}$. It is easily seen from (6.81) and (6.83) that

$$\begin{aligned}\dot{\xi} &= (A + B\overline{F})\overline{x} - (A + BF)\hat{x} = (A + B\overline{F})\xi + B(\overline{F} - F)\hat{x} = \\ &= (A + B\overline{F})\xi + B(D^TD)^{-1}D^T\nu. \end{aligned} \tag{6.85}$$

Furthermore, $\xi(0) = 0$ and $\zeta = \gamma^{-2}E^TX\xi$. Hence

$$\|\nu\|^2 - \gamma^2\|\zeta\|^2 = \int_0^\infty \left(\nu^T\nu - \gamma^{-2}\xi^T XEE^TX\xi\right) dt. \tag{6.86}$$

Since $\hat{x}, \overline{x} \in L_2^n(0,\infty)$, it also holds that $\xi \in L_2^q(0,\infty)$. So, $\nu \in L_2^m(0,\infty)$ and thus $\dot{\xi} \in L_2^n(0,\infty)$. From Lemma 2.1.2 it then follows that $\xi(t) \to 0$ for $t \to \infty$. Together with $\xi(0) = 0$, this implies

$$\int_0^\infty \frac{d}{dt}\xi^T X\xi dt = 0.$$

Subtracting this term from (6.86) and consecutively using (6.85), (6.77), (6.73), (6.74), and (6.75) yields

$$\begin{aligned}
\|\nu\|^2 - \gamma^2\|\zeta\|^2 &= \int_0^\infty \left(\nu^T\nu - \gamma^{-2}\xi^T XEE^T X\xi - \frac{d}{dt}\xi^T X\xi\right) dt = \\
&= \int_0^\infty \Big(\nu^T\nu - 2\nu^T D(D^T D)^{-1} B^T X\xi + \\
&\qquad -\xi^T(\overline{A}^T X + X\overline{A} - 2XSX + \gamma^{-2}XEE^T X)\xi\Big)\, dt = \\
&= \int_0^\infty \Big(\left|\nu - D(D^T D)^{-1} B^T X\xi\right|^2 + \\
&\qquad -\xi^T(\overline{A}^T X + X\overline{A} - XSX + \gamma^{-2}XEE^T X)\xi\Big)\, dt = \\
&= \int_0^\infty \left(\left|\nu - D(D^T D)^{-1} B^T X\xi\right|^2 + \xi^T Q\xi\right) dt. \qquad (6.87)
\end{aligned}$$

This expression is nonnegative if Q is positive semi-definite. The matrix $I - D(D^T D)^{-1} D^T$ is a projection, thus its spectrum is contained in the set $\{0, 1\}$. This implies that Q is positive semi-definite. Hence,

$$J_\gamma(F, \overline{w}, x_0) \geq J_\gamma(\overline{F}, \overline{w}, x_0) \text{ for all } F \in \mathcal{F}. \qquad (6.88)$$

This inequality and the inequality in (6.82) show that $(\overline{F}, \overline{w})$ is a saddle-point solution for the game $(\mathcal{F}, L_2^q(0,\infty), J_\gamma(\cdot,\cdot,x_0))$. Thus, from Lemma 6.3.2 we obtain the following result.

Theorem 6.3.5 *Consider Problem 6.3.1. Assume that D is injective. Let the matrices $Q, \overline{A}$, and S be defined by* (6.72), (6.73), *and* (6.74), *respectively. Assume that the algebraic Riccati equation* (6.75) *has a stabilizing positive semi-definite solution X for some number $\gamma \neq 0$. Define the feedback matrix $\overline{F}$ and the disturbance $\overline{w}$ by*

$$\overline{F} = -(D^T D)^{-1}(D^T C + B^T X); \qquad (6.89)$$

$$\overline{w}(t) = \gamma^{-2} E^T X e^{t(\overline{A} - SX + \gamma^{-2}EE^T X)} x_0 \qquad (6.90)$$

respectively. Then $\overline{F} \in \mathcal{F}$ and $\overline{w} \in L_2^q(0,\infty)$. If $\|\overline{w}\| = r$, then the pair $(\overline{F}, \overline{w})$ is a saddle-point solution for the game $(\mathcal{F}, B_r, J(\cdot,\cdot,x_0))$ and the value of this game equals $x_0^T X x_0 + \gamma^2 r^2$.

The existence of a saddle-point solution $(\overline{F}, \overline{w})$ ensures that lower and upper value are equal, and moreover that the lower and upper value are attained by $\overline{w}$ and $\overline{F}$, respectively. Hence, from Theorem 6.3.5 we can immediately derive the following result.

Corollary 6.3.6 *Assume that the assumptions of Theorem 6.3.5 are fulfilled and let the feedback matrix $\overline{F}$ and the disturbance $\overline{w}$ be as defined in this theorem. Assume that $\|\overline{w}\| = r$. Then*

$$\max_{w\in B_r}\inf_{F\in\mathcal{F}} J(F,w,x_0) = \min_{F\in\mathcal{F}} J(F,\overline{w},x_0) = x_0^T X x_0 + \gamma^2 r^2, \tag{6.91}$$

$$\min_{F\in\mathcal{F}}\sup_{w\in B_r} J(F,w,x_0) = \max_{w\in B_r} J(\overline{F},w,x_0) = x_0^T X x_0 + \gamma^2 r^2. \tag{6.92}$$

It is well-known from H_∞ control theory that under the extra condition that (A, B, C, D) has no invariant zeros on the imaginary axis, the algebraic Riccati equation (6.75) has a positive semi-definite stabilizing solution X if and only if there exists a feedback matrix $F \in \mathcal{F}$ such that $\gamma > \|\mathcal{G}_F\|$ (see for instance [128, Theorem 16.4] or [119, Theorem 8.12.1]). Furthermore, if this holds, one such feedback matrix is the matrix $-(D^TD)^{-1}(C^TD+B^TX)$. Using this standard H_∞ result, Theorem 6.3.5 can straightforwardly be further specified as follows.

Theorem 6.3.7 *Consider Problem 6.3.1. Assume that D is injective and that (A, B, C, D) has no invariant zeros on the imaginary axis. Let γ^* be defined by* (6.63), *and let the matrices Q, $\overline{A}$, and S be defined by* (6.72), (6.73), *and* (6.74), *respectively. Then, the algebraic Riccati equation* (6.75) *has a stabilizing positive semi-definite solution if and only if $\gamma > \gamma^*$. Denote this solution by X_γ. Define for each $\gamma > \gamma^*$ the feedback matrix F_γ and the disturbance w_γ by*

$$F_\gamma = -(D^TD)^{-1}(D^TC + B^TX_\gamma); \tag{6.93}$$

$$w_\gamma(t) = \gamma^{-2}E^TX_\gamma e^{t(\overline{A}-SX_\gamma+\gamma^{-2}EE^TX_\gamma)}x_0. \tag{6.94}$$

Then $F_\gamma \in \mathcal{F}$ and $w_\gamma \in L_2^q(0,\infty)$. If $\|w_\gamma\| = r$, then the pair (F_γ, w_γ) is a saddle-point solution for the game $(\mathcal{F}, B_r, J(\cdot,\cdot,x_0))$ and the value of this game equals $x_0^TX_\gamma x_0 + \gamma^2r^2$.

Proof By definition of γ^*, we have $\gamma > \gamma^*$ if an only if there exists a state feedback matrix $F \in \mathcal{F}$ such that $\gamma^* \le \|\mathcal{G}_F\| < \gamma$. Hence, according to the statement following Corollary 6.3.6 the algebraic Riccati equation (6.75) has a positive semi-definite stabilizing solution if and only if $\gamma > \gamma^*$. The rest of the theorem follows by applying Theorem 6.3.5 with $\overline{F} = F_\gamma$ and $\overline{w} = w_\gamma$. □

Assume that the assumptions of Theorem 6.3.7 are fulfilled and let X_γ, F_γ and w_γ be as in this theorem. We proceed by investigating whether a $\gamma > \gamma^*$ exists such that $\|w_\gamma\| = r$. The next lemma states that $\|w_\gamma\|$ is decreasing in γ. This lemma is similar to Lemma 6.2.11 (or the second statement of Lemma 6.2.17). In that case it was shown that the norm of the worst-case disturbance is decreasing in γ which also holds in the present case in the following sense. Given that the system is controlled by the feedback matrix F_γ, the norm of the worst-case disturbance is decreasing in γ.

Lemma 6.3.8 *Let $\gamma^* < \gamma_1 < \gamma_2$. Then $\|w_{\gamma_1}\| \geq \|w_{\gamma_2}\|$.*

Proof The pair (F_γ, w_γ) is a saddle-point solution for the soft-constrained game. Thus the following inequalities hold for all $w \in L_2^q(0, \infty)$ and for all $F \in \mathcal{F}$:

$$\begin{aligned} J(F_\gamma, w, x_0) - \gamma^2\|w\|^2 &\leq J(F_\gamma, w_\gamma, x_0) - \gamma^2\|w_\gamma\|^2, \\ J(F_\gamma, w_\gamma, x_0) &\leq J(F, w_\gamma, x_0). \end{aligned} \tag{6.95}$$

Consecutively using the second inequality with $\gamma = \gamma_1$, $F = F_{\gamma_2}$, and the first inequality with $\gamma = \gamma_2$, $w = w_{\gamma_1}$ yields

$$\begin{aligned} J(F_{\gamma_1}, w_{\gamma_1}, x_0) \leq J(F_{\gamma_2}, w_{\gamma_1}, x_0) &= J(F_{\gamma_2}, w_{\gamma_1}, x_0) - \gamma_2^2\|w_{\gamma_1}\|^2 + \gamma_2^2\|w_{\gamma_1}\|^2 \leq \\ &\leq J(F_{\gamma_2}, w_{\gamma_2}, x_0) - \gamma_2^2\|w_{\gamma_2}\|^2 + \gamma_2^2\|w_{\gamma_1}\|^2, \end{aligned} \tag{6.96}$$

or, equivalently,

$$J(F_{\gamma_1}, w_{\gamma_1}, x_0) - J(F_{\gamma_2}, w_{\gamma_2}, x_0) \leq \gamma_2^2 \left(\|w_{\gamma_1}\|^2 - \|w_{\gamma_2}\|^2\right). \tag{6.97}$$

The assumption $\gamma_1 < \gamma_2$ has not yet been used, so the roles of γ_1 and γ_2 can be interchanged in this inequality. Hence

$$J(F_{\gamma_2}, w_{\gamma_2}, x_0) - J(F_{\gamma_1}, w_{\gamma_1}, x_0) \leq \gamma_1^2 \left(\|w_{\gamma_2}\|^2 - \|w_{\gamma_1}\|^2\right). \tag{6.98}$$

Adding the inequalities (6.97) and (6.98) produces the inequality

$$\gamma_1^2 \left(\|w_{\gamma_1}\|^2 - \|w_{\gamma_2}\|^2\right) \leq \gamma_2^2 \left(\|w_{\gamma_1}\|^2 - \|w_{\gamma_2}\|^2\right),$$

which can equivalently be written as

$$\left(\gamma_1^2 - \gamma_2^2\right)\left(\|w_{\gamma_1}\|^2 - \|w_{\gamma_2}\|^2\right) \leq 0.$$

Since $\gamma_1 < \gamma_2$, this inequality shows that $\|w_{\gamma_1}\| \geq \|w_{\gamma_2}\|$. □

Remark 6.3.9 Recall from (6.82) that the inequality (6.95) is strict if $w \neq w_\gamma$. This implies that (6.96) is strict if $w_{\gamma_1} \neq w_{\gamma_2}$. Hence, if $\gamma \mapsto w_\gamma$ is injective, the monotonicity of $\|w_\gamma\|$ is strict. □

Consider the limit $\gamma \to \infty$. Since (A, B) is stabilizable and (A, B, C, D) has no invariant zeros on the imaginary axis, the stabilizing solution of the algebraic Riccati equation

$$Q + \overline{A}^T X + X\overline{A} - XSX = 0 \tag{6.99}$$

exists. Denote this solution by X_∞. Due to this existence, the stabilizing solution X_γ of (6.75) converges to X_∞ for $\gamma \to \infty$. Consequently,

$$\gamma^2 w_\gamma \to w_\infty, \text{ for } \gamma \to \infty. \tag{6.100}$$

where w_∞ is defined by

$$w_\infty(t) := E^T X_\infty e^{t(A - B(D^T D)^{-1}(C^T D + B^T X_\infty))} x_0. \tag{6.101}$$

Note that $w_\infty \in L_2^q(0, \infty)$ since X_∞ is the stabilizing solution of (6.99). We obtain

$$\|w_\gamma\| \downarrow 0 \text{ for } \gamma \to \infty. \tag{6.102}$$

Next, consider the limit $\gamma \downarrow \gamma^*$. It is well-known that γ^* equals the infimum of all numbers γ for which the algebraic Riccati equation (6.75) has a positive semi-definite stabilizing solution [128, Theorem 16.4]. So, if $\gamma = \gamma^*$, either the stabilizing solution does not exist or it does exist but it is not positive semi-definite. However, the latter situation is not possible because X_γ is a decreasing function of γ. Hence, if $\gamma = \gamma^*$, the stabilizing solution does not exist. By definition, we know that

$$\operatorname{im} \begin{bmatrix} I \\ X_\gamma \end{bmatrix} \tag{6.103}$$

equals the spectral subspace of the Hamiltonian matrix

$$H_\gamma := \begin{bmatrix} \overline{A} & -S + \gamma^{-2} E E^T \\ -Q & -\overline{A}^T \end{bmatrix}, \tag{6.104}$$

corresponding to the eigenvalues in the open left-half plane. Using this observation it can be argued along the same lines as in the discussion preceding Theorem 6.2.13 that generically $\|w_\gamma\|$ diverges to infinity for $\gamma \downarrow \gamma^*$. We shall not go into the problem of finding exact conditions under which this happens. An illustrative example about the limiting behavior of the H_∞ suboptimal solution can be found in [128, Section 16.9]. We summarize the main result of the present section in the following theorem.

Theorem 6.3.10 *Consider Problem 6.3.1. Assume that the conditions of Theorem 6.3.7 are fulfilled and let X_γ, F_γ and w_γ be as in this theorem. If $r < \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$, a number $\hat{\gamma} > \gamma^*$ exists such that $\|w_{\hat{\gamma}}\| = r$. In this case, the pair $(F_{\hat{\gamma}}, w_{\hat{\gamma}})$ is a saddle-point solution for the game $(\mathcal{F}, B_r, J(\cdot, \cdot, x_0))$ and the value of this game equals $x_0^T X_{\hat{\gamma}} x_0 + \hat{\gamma}^2 r^2$.*

Due to the monotonic behavior of $\|w_\gamma\|$, it is in principle straightforward to construct a numerical scheme producing the number $\hat{\gamma}$ (if it exists). Indeed, one starts with a sufficiently large value of γ and determines the corresponding positive semi-definite solution of the algebraic Riccati equation (6.75). Next, $\|w_\gamma\|$ can be computed and depending on whether the result is smaller or larger than r^2, the procedure is repeated with an updated value of γ. The monotonicity of $\|w_\gamma\|$ ensures that the sequence of updated values of γ converges to $\hat{\gamma}$.

6.4 Risk-Sensitive Nash Equilibria

The theory developed in the previous sections is applicable to N-player infinite-horizon stable deterministically disturbed linear time-invariant input-output differential games, where the information structure of the players is a memoryless perfect state pattern and where they are restricted to linear time-invariant strategies. In this section we consider such a game where the cost function of player i is represented by the L_2-norm of his output. See Section 2.3 for the terminology and the corresponding notations. We shall define a naturally corresponding equilibrium concept, and derive a system of coupled equations generating such equilibria under certain assumptions. This system will be studied in more detail for the simplified scalar case.

The cost function of player i is the square of the L_2-norm of his output, i.e. the functional J_i : $\mathcal{F}_N \times L_2^q(0, \infty) \times I\!R^n \to I\!R$ defined by

$$J_i(F_1, \ldots, F_N, w, x_0) = \|z_i\|^2. \tag{6.105}$$

The outputs z_i are determined by the system equations

$$\dot{x} = \left(A + \sum_{j=1}^{N} B_j F_j\right) x + Ew, \quad x(0) = x_0, \tag{6.106}$$

$$z_i = \left(C_i + \sum_{j=1}^{N} D_{ij} F_j\right) x, \quad i = 1, \ldots, N. \tag{6.107}$$

We assume that each player has his own belief about how big the disturbance is and that he looks for a strategy minimizing his cost function corrupted by a bounded worst-case disturbance, i.e.

player i aims to minimize

$$\overline{J}_i(F_1, \dots, F_N, r_i, x_0) := \sup_{\|w\| \le r_i} J_i(F_1, \dots, F_N, w, x_0), \tag{6.108}$$

where the number $r_i > 0$ expresses player i's belief about the maximal size of the disturbance. This number is called the *risk-sensitivity parameter* of player i. If r_i is small, player i has relatively low costs but his criterion incorporates only disturbances with a small size, i.e. he is not risk-averse. For larger values of r_i, player i is more risk-averse, which results in larger costs. Using the functionals $\overline{J}_i$ as adjusted cost functions we define the following robust equilibrium concept for the disturbed differential game under consideration.

Definition 6.4.1 An N-tuple $\overline{F} = (\overline{F}_1, \dots, \overline{F}_N) \in \mathcal{F}_N$ of feedback matrices is called a *risk-sensitive memoryless perfect state Nash equilibrium* with respect to the risk-sensitivity parameters $r_1, \dots, r_N$, if for all i the following inequality holds:

$$\overline{J}_i(\overline{F}, r_i, x_0) \le \overline{J}_i(\overline{F}_{-i}(F_i), r_i, x_0) \tag{6.109}$$

for all $F_i \in I\!R^{m_i \times n}$ such that $\overline{F}_{-i}(F_i) \in \mathcal{F}_N$. □

Remark 6.4.2 An N-tuple $(\overline{F}_1, \dots, \overline{F}_N) \in \mathcal{F}_N$ satisfying $C_i + \sum_{j=1}^N D_{ij}\overline{F}_j = 0$ for all $i = 1, \dots, N$ is clearly a risk-sensitive memoryless perfect state Nash equilibrium with respect to any set of risk-sensitivity parameters; each player has zero costs in this equilibrium. We shall refer to such an equilibrium as a *trivial equilibrium*. If $D_{ij} = 0$ for all $i \neq j$ and $D_i := D_{ii}$ is injective for all i, then a trivial equilibrium exists if and only if (i) each column of C_i is contained in $\operatorname{im} D_i$ and (ii) the matrix

$$\tilde{A} := A - \sum_{j=1}^N (D_j^T D_j)^{-1} D_j^T C_j \tag{6.110}$$

is stable. In that case, the trivial equilibrium is uniquely given by $\overline{F}_i = -(D_i^T D_i)^{-1} D_i^T C_i$. This situation can for example be excluded by assuming that the matrix $\tilde{A}$ has at least one eigenvalue in the open right-half plane, sometimes referred to as the *nonminimum phase condition*. □

We shall apply Corollary 6.3.6 to obtain a system of equations, from which risk-sensitive memoryless perfect state Nash equilibria can be determined. To that end, let $\overline{F}_1, \dots, \overline{F}_{i-1}, \overline{F}_{i+1}, \dots, \overline{F}_N$ be fixed feedback matrices for all the players except player i. Consider the situation where player i aims to minimize

$$\overline{J}_i(\overline{F}_1, \dots, \overline{F}_{i-1}, F_i, \overline{F}_{i+1}, \dots, \overline{F}_N, r_i, x_0) \tag{6.111}$$

by choosing an appropriate feedback matrix F_i in the set

$$\hat{\mathcal{F}} := \left\{ F \in I\!R^{m_i \times n} \;\middle|\; A + \sum_{j \neq i}^{N} B_j \overline{F}_j + B_i F \text{ is stable} \right\}. \tag{6.112}$$

This minimization problem for player i is equivalent to the minmax problem

$$\min_{F_i \in \hat{\mathcal{F}}} \sup_{w \in B_{r_i}} J_i(\overline{F}_1, \ldots, \overline{F}_{i-1}, F_i, \overline{F}_{i+1}, \ldots, \overline{F}_N) \tag{6.113}$$

corresponding to the linear system

$$\begin{aligned} \dot{x} &= \left(A + \sum_{j \neq i}^{N} B_j \hat{F}_j + B_i F_i \right) x + Ew, \quad x(0) = x_0, \\ z_i &= \left(C_i + \sum_{j \neq i}^{N} D_{ij} \hat{F}_j + D_{ii} F_i \right) x. \end{aligned}$$

A solution of this problem can be obtained from Corollary 6.3.6; specifically from (6.92). Indeed, assume that $D_i := D_{ii}$ is injective and define the matrices

$$Q_i := C_i^T (I - D_i (D_i^T D_i)^{-1} D_i^T) C_i, \tag{6.114}$$
$$\overline{A}_i := A - B_i (D_i^T D_i)^{-1} D_i^T C_i, \tag{6.115}$$
$$S_i := B_i (D_i^T D_i)^{-1} B_i^T. \tag{6.116}$$

Furthermore, assume that there exists a positive semi-definite $n \times n$ matrix X_i and a number $\gamma_i \neq 0$ such that

$$\begin{aligned} &\left(C_i + \sum_{j \neq i}^{N} D_{ij} \overline{F}_j \right)^T (I - D_i (D_i^T D_i)^{-1} D_i^T) \left(C_i + \sum_{j \neq i}^{N} D_{ij} \overline{F}_j \right) + \\ &\quad + \left(\overline{A}_i + \sum_{j \neq i}^{N} B_j \overline{F}_j - B_i (D_i^T D_i)^{-1} D_i^T \sum_{j \neq i}^{N} D_{ij} \overline{F}_j \right)^T X_i + \\ &\quad + X_i \left(\overline{A}_i + \sum_{j \neq i}^{N} B_j \overline{F}_j - B_i (D_i^T D_i)^{-1} D_i^T \sum_{j \neq i}^{N} D_{ij} \overline{F}_j \right) + \\ &\quad - X_i S_i X_i + \gamma_i^{-2} X_i E E^T X_i = 0, \end{aligned} \tag{6.117}$$

and

$$\overline{A}_i + \sum_{j \neq i}^{N} B_j \overline{F}_j - B_i (D_i^T D_i)^{-1} D_i^T \sum_{j \neq i}^{N} D_{ij} \overline{F}_j - S_i X_i + \gamma_i^{-2} E E^T X_i \text{ is stable}. \tag{6.118}$$

Now, define the feedback matrix $\overline{F}_i$ and the disturbance $\overline{w}_i$ by

$$\overline{F}_i := -(D_i^T D_i)^{-1} \left(D_i^T \left(C_i + \sum_{j \neq i}^{N} D_{ij} \overline{F}_j \right) + B_i^T X_i \right), \tag{6.119}$$

$$\overline{w}_i(t) := \gamma_i^{-2} E^T X_i \exp \left(t \left(\overline{A}_i + \sum_{j \neq i}^{N} B_j \overline{F}_j - B_i (D_i^T D_i)^{-1} D_i \sum_{j \neq i}^{N} D_{ij} \overline{F}_j + \right.\right.$$
$$\left.\left. - S_i X_i + \gamma_i^{-2} E E^T X_i \right) \right) x_0. \tag{6.120}$$

Note that $\overline{F}_i \in \hat{\mathcal{F}}$. According to Corollary 6.3.6, if $\|\overline{w}_i\| = r_i$ it follows that $\overline{F}_i$ solves the minmax problem (6.113) for player i.

The feedback matrix $\overline{F}_i$ is a best reply matrix for player i against the given feedback matrices $\overline{F}_1, \ldots, \overline{F}_{i-1}, \overline{F}_{i+1}, \ldots, \overline{F}_N$ of the other players. In this way, given that all the involved assumptions hold, a best reply matrix can be found for each player. Any intersection point of these best reply matrices is a robust memoryless perfect state Nash equilibrium. To simplify matters, let us further assume that $D_{ij} = 0$ for all $i \neq j$. Let the matrix $\tilde{A}$ be defined by (6.110). Then, it is easily verified that if the N-tuple $(X_1, \ldots, X_N)$ of symmetric positive semi-definite $n \times n$ matrices and the N-tuple of nonzero numbers $(\gamma_1, \ldots, \gamma_N)$ satisfy for all $i = 1, \ldots, N$ the equation

$$Q_i + \tilde{A}^T X_i + X_i \tilde{A} - \sum_{j \neq i}^{N} (X_i S_j X_j + X_j S_j X_i) - X_i S_i X_i + \gamma_i^{-2} X_i E E^T X_i = 0$$

such that

$$A_i := \tilde{A} - \sum_{j=1}^{N} S_j X_j + \gamma_i^{-2} E E^T X_i \text{ is stable for each } i = 1, \ldots, N$$

and if $\|w_i\| = r_i$ with

$$w_i(t) := \gamma_i^{-2} E^T X_i e^{t A_i} x_0$$

then the N-tuple $(\overline{F}_1, \ldots, \overline{F}_N)$ with

$$\overline{F}_i := -(D_i^T D_i)^{-1} (D_i^T C_i + B_i^T X_i)$$

is an intersection point of best reply matrices. The condition $\|w_i\| = r_i$ can equivalently be written as the equation $\gamma_i^{-4} x_0^T P_i x_0 = r_i^2$ with P_i the solution of the Lyapunov equation $A_i^T P_i + P_i A_i = -X_i E E^T X_i$. We formulate the obtained system of equations generating risk-sensitive memoryless perfect state Nash equilibria in the following theorem.

Theorem 6.4.3 *Consider an N-player infinite-horizon stable deterministically disturbed linear time-invariant input-output differential game, where the information structure of the players is a memoryless perfect state pattern and where they are restricted to linear time-invariant strategies. Let the cost function of player i be the square of the L_2-norm of his output. Assume that $D_{ij} = 0$ for all $i \neq j$, and that $D_i := D_{ii}$ is injective for all $i = 1, \dots, N$. Let r_i be the risk-sensitivity parameter of player i. Let the matrices Q_i and S_i be defined by* (6.114) *and* (6.116)*, respectively. Furthermore, define*

$$\tilde{A} := A - \sum_{j=1}^{N} B_j (D_j^T D_j)^{-1} D_j^T C_j. \tag{6.121}$$

Assume there exist N symmetric $n \times n$ matrices X_i, N symmetric $n \times n$ matrices P_i, and N nonzero real numbers γ_i, such that

$$X_i \geq 0; \tag{6.122}$$

$$Q_i + \tilde{A}^T X_i + X_i \tilde{A} - \sum_{j \neq i}^{N} (X_i S_j X_j + X_j S_j X_i) - X_i S_i X_i + \gamma_i^{-2} X_i E E^T X_i = 0; \tag{6.123}$$

$$A_i := \tilde{A} - \sum_{j=1}^{N} S_j X_j + \gamma_i^{-2} E E^T X_i \text{ is stable for each } i = 1, \dots, N; \tag{6.124}$$

$$A_i^T P_i + P_i A_i = -X_i E E^T X_i; \tag{6.125}$$

$$\gamma_i^{-4} x_0^T P_i x_0 = r_i^2. \tag{6.126}$$

Then the N-tuple of feedback matrices $(\overline{F}_1, \dots, \overline{F}_N)$ with

$$\overline{F}_i := -(D_i^T D_i)^{-1} (D_i^T C_i + B_i^T X_i) \tag{6.127}$$

is a risk-sensitive memoryless perfect state Nash equilibrium with respect to the risk-sensitivity parameters $r_1, \dots, r_N$ and

$$\overline{J}_i(\overline{F}_1, \dots, \overline{F}_N, r_i, x_0) = x_0^T X_i x_0 + \gamma_i^2 r_i^2. \tag{6.128}$$

Note that if $E = 0$, equation (6.123) is exactly the i-th equation in the system of algebraic Riccati equations (5.13), the stabilizing solutions of which characterize deterministic feedback Nash equilibria (see Theorem 5.3.2). The matrix A in (5.13) has been replaced by the matrix $\tilde{A}$ in (6.123) which is due to the input-output structure considered in the present chapter. The positivity requirement (6.122) is not required in Theorem 5.3.2 since we considered indefinite cost functions in the previous chapter. Here we consider positive cost criteria which results in the requirement (6.122). As a result, closed-loop stability is automatically fulfilled. This in contrast to the indefinite case where closed-loop stability is an explicit requirement. Here we deal with different type of stability requirements, i.e. the N conditions (6.124). These conditions ensure that the Lyapunov equations (6.125) are uniquely solvable. The solvability of the last equations (6.126) is however unclear; in the one-player case it was shown that the left-hand side of (6.126) is decreasing in γ for $\gamma > \gamma^*$ (Lemma 6.3.8). In the N-player case, we deal with a coupled system of N nonlinear equations in the N unknowns $\gamma_1, \ldots, \gamma_N$ which need to be solved in a set Γ, defined as the collection of N-tuples $(\gamma_1, \ldots, \gamma_N)$ of nonzero real numbers with the property that there exists an N-tuple $X_1, \ldots, X_N$ satisfying (6.122)-(6.124).

In the rest of this section we assume that $D_{ij} = 0$ for $i \neq j$ and that $D_i := D_{ii}$ is injective for all i and we consider the simplified two-player scalar case in order to obtain insight in the solvability of the system (6.122)-(6.126). This is a substantial simplification since the matrix Q_i becomes zero for each i. This is due to the input-output structure considered in this chapter. This in contrast to the "(A, B, Q, R)" setting considered in Chapters 5 and 7. It shows that the input-output setting is mathematically more structured.

The unknowns P_i can immediately be eliminated from the system (6.122)-(6.126). Denote $x_i := X_i$, $\tilde{a} := \tilde{A}$, $s_i := S_i$, and $e = E$, then this system reduces to

$$x_1, x_2 \geq 0; \tag{6.129}$$

$$2\tilde{a}x_1 - 2s_2x_1x_2 + \left(\frac{e^2}{\gamma_1^2} - s_1\right)x_1^2 = 0; \tag{6.130}$$

$$2\tilde{a}x_2 - 2s_1x_1x_2 + \left(\frac{e^2}{\gamma_2^2} - s_2\right)x_2^2 = 0; \tag{6.131}$$

$$\tilde{a} + \left(\frac{e^2}{\gamma_1^2} - s_1\right)x_1 - s_2x_2 < 0; \tag{6.132}$$

$$\tilde{a} - s_1x_1 + \left(\frac{e^2}{\gamma_2^2} - s_2\right)x_2 < 0; \tag{6.133}$$

$$x_1^2 = -\frac{2r_1^2}{x_0^2e^2}\left(\tilde{a} + \left(\frac{e^2}{\gamma_1^2} - s_1\right)x_1 - s_2x_2\right)\gamma_1^4; \tag{6.134}$$

$$x_2^2 = -\frac{2r_2^2}{x_0^2e^2}\left(\tilde{a} - s_1x_1 + \left(\frac{e^2}{\gamma_2^2} - s_1\right)x_2\right)\gamma_2^4. \tag{6.135}$$

Motivated by Remark 6.4.2 we exclude the existence of the trivial equilibrium by assuming that

$$\tilde{a} > 0.$$

System (6.129)-(6.135) should be interpreted as follows. Conditions (6.129)-(6.133) define a region $\Gamma \subset I\!R_+^2$ of pairs (γ_1, γ_2) such that a solution (x_1, x_2) of (6.129)-(6.133) exists if and only if $(\gamma_1, \gamma_2) \in \Gamma$. By substituting such a solution in equations (6.134) and (6.135) a system of two nonlinear equations in γ_1 and γ_2 results. A solution of this system in Γ defines a risk-sensitive memoryless perfect state Nash equilibrium via (6.127). Instead of working with the variables x_1, x_2 and γ_1, γ_2 it is more convenient to apply the decoupled coordinate transformations

$$\xi_i := s_i x_i; \tag{6.136}$$

$$\omega_i := \frac{e^2}{s_i \gamma_i^2} - 1 \tag{6.137}$$

which are regular transformations if and only if $s_i \neq 0$. Under these transformations, the system (6.129)-(6.135) changes into

$$\xi_1, \xi_2 \geq 0; \tag{6.138}$$

$$2\tilde{a}\xi_1 - 2\xi_1\xi_2 + \omega_1\xi_1^2 = 0; \qquad 2\tilde{a}\xi_2 - 2\xi_1\xi_2 + \omega_2\xi_2^2 = 0; \tag{6.139}$$

$$\tilde{a} + \omega_1\xi_1 - \xi_2 < 0; \qquad \tilde{a} - \xi_1 + \omega_2\xi_2 < 0; \tag{6.140}$$

$$\xi_1^2(\omega_1 + 1)^2 = \rho_1(\tilde{a} + \omega_1\xi_1 - \xi_2); \quad \xi_2^2(\omega_2 + 1)^2 = \rho_2(\tilde{a} - \xi_1 + \omega_2\xi_1) \tag{6.141}$$

where we have defined the numbers

$$\rho_i := -\frac{2r_1^2e^2}{x_0^2}, \quad i = 1, 2. \tag{6.142}$$

Instead of determining the set Γ we shall determine the set Ω which is defined in such a way that a solution (ξ_1, ξ_2) of (6.138)-(6.140) exists if and only if $(\omega_1, \omega_2) \in \Omega$. From (6.137) we see that Ω should be contained in the set $\{(\omega_1, \omega_2) | \omega_1, \omega_2 > -1\}$. Note that in the limit case $\omega_1 = \omega_2 = -1$, which corresponds with $\gamma_1 = \gamma_2 = \infty$ (the undisturbed case), equations (6.139) agree with the hyperbola equations (5.36) and (5.37) in the degenerate case $\sigma_1 = \sigma_2 = 0$. In this case, the hyperbolas are just a pair of lines, i.e. the asymptotes. In the disturbed case $\omega_{1,2} > -1$ the equations for the two pairs of lines are

$$\xi_1 = 0, \quad \xi_2 = (\omega_1/2)\xi_1 + \tilde{a}; \tag{6.143}$$

$$\xi_2 = 0, \quad \xi_1 = (\omega_2/2)\xi_2 + \tilde{a}. \tag{6.144}$$

The first of the pair of lines described by (6.143) coincides with the ξ_2-axis and the second intersects the ξ_2-axis at $\xi_2 = \tilde{a}$ under an angle between $-\arctan 1/2$ and $\pi/2$ depending on the value of ω_1. Similarly, the first of the pair of lines described by (6.144) coincides with the ξ_1-axis and the second intersects the ξ_1-axis at $\xi_1 = \tilde{a}$ under an angle between zero and $\arctan 1/2 + \pi/2$ depending on the value of ω_2. We are interested in each intersection point of the two pairs satisfying (6.138) and (6.140). Furthermore, for each intersection point ξ^i we want to determine the set Ω^i defined as the collection of points (ω_1, ω_2) with $\omega_i > -1$ and such that ξ^i satisfies (6.138) and (6.140). The set Ω is then the union of these sets Ω_i. It is easily seen that zero, one or three intersection points of the two pairs of lines satisfying (6.138) exist, i.e.

$$\begin{aligned} \xi^1 &= \left(0, -\frac{2\tilde{a}}{\omega_2}\right) && \Leftrightarrow \quad \omega_2 < 0; \\ \xi^2 &= \left(-\frac{2\tilde{a}}{\omega_1}, 0\right) && \Leftrightarrow \quad \omega_1 < 0; \\ \xi^3 &= \frac{2\tilde{a}}{4 - \omega_1\omega_2}(\omega_2 + 2, \omega_1 + 2) && \Leftrightarrow \quad \omega_1\omega_2 < 4. \end{aligned}$$

The intersection points ξ^1 and ξ^2 also satisfy the stability requirements (6.140) under the conditions $\omega_2 < 0$ and $\omega_1 < 0$, respectively. However, the third intersection point ξ^3 satisfies these stability requirements if and only if $\omega_{1,2} < 0$. Under this condition the condition $\omega_1\omega_2 < 4$ also holds. We conclude that

$$\begin{aligned} \Omega^1 &= \{(\omega_1, \omega_2) \in I\!R^2 | \omega_1 > -1, \quad -1 < \omega_2 < 0\}; \\ \Omega^2 &= \{(\omega_1, \omega_2) \in I\!R^2 | -1 < \omega_1 < 0, \quad -1 < \omega_2\}; \\ \Omega^3 &= \{(\omega_1, \omega_2) \in I\!R^2 | -1 < \omega_1 < 0, \quad -1 < \omega_2 < 0\}. \end{aligned}$$

If only one of the numbers ω is negative, the system (6.138)-(6.140) has one solution; if they are both negative, this system has three different solutions. The next step is to substitute for each $(\omega_1, \omega_2) \in \Omega^i$ the corresponding solution ξ^i in (6.141) and to solve the resulting equations in Ω^i. It is immediately clear that substituting ξ^1 in the first equation of (6.141) results in an equation with a zero left-hand side and a nonzero right-hand side for each $(\omega_1, \omega_2) \in \Omega^1$. Hence the resulting equation is unsolvable in Ω^1. For similar reasons, substituting ξ^2 in the second equation of (6.141) leads to an unsolvable equation in Ω^2. Thus in the present case intersection point ξ^3 is the only possibility to obtain a risk-sensitive memoryless perfect state Nash equilibrium from Theorem 6.4.3. Substituting ξ^3 in the equations (6.141) results in the following system of two equations:

$$4\tilde{a}(\omega_1 + 1)^2(\omega_2 + 2) = \rho_1\omega_1(4 - \omega_1\omega_2); \tag{6.145}$$

$$4\tilde{a}(\omega_2 + 1)^2(\omega_1 + 2) = \rho_2\omega_2(4 - \omega_1\omega_2). \tag{6.146}$$

Equation (6.145) is linear in ω_2. Thus from this equation ω_2 can be determined as a function of ω_1. This function is a quotient of two quadratic polynomials, which can be substituted in equation (6.146). This results in a fifth-order polynomial equation in ω_1. At this point analytic computations seem to end; of course one can proceed straightforwardly in concrete situations.

By considering the two-player scalar case, the complexity of the set of conditions given in Theorem 6.4.3 has become clear. In general, numerical techniques are required to obtain solutions from this system. More theoretical research is required to obtain structural results about the solvability of the nonlinear equations in the unknowns γ_i in the set Γ. One can easily construct scalar examples with three risk-sensitive memoryless perfect state Nash equilibria where the set of sufficient conditions as given in Theorem 6.4.3 only describes one of them. This shows the need of necessary and sufficient conditions characterizing all risk-sensitive memoryless perfect state Nash equilibria. Such conditions have been found in the deterministic case in Chapter 5. It is a challenge to see whether necessary and sufficient conditions can be found here along the same lines.

6.5 Concluding Remarks

An overview of the problems studied and the results obtained in the Sections 6.2 and 6.3 can be found in the Sections 8.2.2 and 8.2.3. Concluding remarks on risk-sensitive Nash equilibria are contained in Section 8.1.4. This section also discusses the relation between these equilibria and other robust equilibria introduced in this thesis. Section 6.3 has been presented at the *Ninth International Symposium on Dynamic Games and Applications* [28]. A preliminary version of Section 6.4 has been presented at the same conference [26].

Chapter 7

Robust Equilibria with Unbounded Disturbances

7.1 Introduction

The soft-constrained differential game ([9], see also Section 6.3 of the present thesis) has been introduced in the literature in order to solve the disturbance attenuation (or H_∞ control) problem using zero-sum game theory. The criterion in the disturbance attenuation problem, i.e. the H_∞ norm, is in terms of a supremum taken over norm bounded disturbances. The soft-constrained differential game criterion has an additional negatively weighted explicit quadratic disturbance term and furthermore, the supremum is taken over all disturbances without requiring the norm to be bounded. In Chapter 6 we studied the disturbance attenuation problem; in contrast with the existing literature, we studied this problem with known nonzero initial state. This setting has been generalized to the N-player case, i.e. in 6.4 we introduced the concept of risk-sensitive memoryless perfect state equilibria in the context of deterministically disturbed differential games with norm bounds on the disturbance. To make the picture complete, we study in the present chapter deterministically disturbed differential games without norm bounds; instead we add negative terms to the cost criteria of the players. Furthermore, the information structure of the players is assumed to be a feedback pattern. Clearly, the soft-constrained differential game is the basis for such a setting.

In Section 7.2 we consider the one-player case, i.e. we shall only be concerned with the soft-constrained differential game. Instead of an input-output formulation as in Chapter 6, which leads automatically to a positive semi-definite state weighting matrix in the cost functional, we consider here a cost criterion without assuming the state weighting matrix to be positive semi-definite, i.e. we consider an indefinite soft-constrained differential game. In this respect, which is in line with the indefinite setting of Chapter 5, we differ from approaches usually taken; see for instance [9, 50, 128], [10, Section 6.6], or [76, Section 20.2]. For that reason the derivation of a saddle-point solution in Section 7.2 is self-contained; links with the analysis in Section 6.3 are obvious. A set of sufficient conditions for the existence of this saddle-point solution involving an algebraic Riccati equation and inequality will be found. The appearance of the Riccati matrix inequality is due to the possible indefiniteness of the state weighting matrix. Section 7.2 ends with a study about the necessity of the sufficient conditions. It turns out that under a restriction of the admissible set of state feedback matrices, the algebraic Riccati equation is a necessary condition for the solvability of the minmax problem.

In Section 7.3 the setting of an N-player deterministically disturbed differential game with soft-constrained cost criteria is introduced. We shall define an equilibrium concept called the soft-constrained feedback Nash equilibrium and the sufficiency result of Section 7.2 can immediately be applied to formulate a set of sufficient conditions in terms of systems of algebraic Riccati equations and inequalities generating such equilibria. The two-player case is worked out in more detail. Surprisingly, the algebraic Riccati inequalities are automatically satisfied in this case.

It is well-known that there is a connection between the H_∞ control problem and the risk sensitive LQG (or LEQG) control problem. (see for instance [62, 53] or [9, Section 4.7]). Consequently, the soft-constrained feedback Nash equilibrium concept can easily be interpreted stochastically by considering LEQG cost criteria. This connection will be made in Section 7.4 for which we use the infinite horizon results from Runolfsson [107]. Games with exponential-of-integral cost criteria are referred to as risk-sensitive dynamic games. Such games were recently studied in a finite-horizon discrete-time linear-quadratic context in [68].

7.2 Indefinite Soft-Constrained Differential Games

In this section we study the one-player case, i.e. we consider a linear system

$$\dot{x} = (A + BF)x + Ew, \quad x(0) = x_0, \tag{7.1}$$

with (A, B) stabilizable. The control feedback matrix is assumed to be in the set $\mathcal{F}$. By definition, the stabilizability of (A, B) ensures that $\mathcal{F} \neq \emptyset$. The disturbance w is assumed to be an element of the Hilbert space $L_2^q(0, \infty)$. The soft-constrained cost criterion is the functional

$J : \mathcal{F} \times L_2^q(0,\infty) \times I\!R^n \to I\!R$, defined by

$$J(F,w,x_0) = \int_0^\infty \left(x^T(Q + F^T RF)x - w^T V w \right) dt. \tag{7.2}$$

The matrices Q, R and V are symmetric, $R > 0$, and $V > 0$. The soft-constrained differential game is the situation where the controller designer is minimizing the criterion (7.2) by choosing an appropriate $F \in \mathcal{F}$, while the uncertainty is maximizing the same criterion by choosing an appropriate $w \in L_2^q(0,\infty)$. Since we do not assume the matrix Q to be positive semi-definite we refer to this game as the *indefinite soft-constrained differential game*. Having the generalization to the N-player case in mind, we are interested in the upper value of this game. To be precise, we study the following problem.

Problem 7.2.1 Determine, if it exists, for each $x_0 \in I\!R^n$ the value

$$\min_{F \in \mathcal{F}} \sup_{w \in L_2^q(0,\infty)} J(F,w,x_0). \tag{7.3}$$

Furthermore, if this value exists, determine a matrix $F \in \mathcal{F}$ attaining this value. □

Remark 7.2.2 If Q is positive semi-definite and if $V = \gamma^2 I$ for some $\gamma \neq 0$, (7.3) equals the upper value of the soft-constrained differential game as considered by Başar and Bernhard [9] with a feedback information pattern for the minimizing player and an open-loop information pattern for the maximizing player. Under the condition that $(A, Q^{1/2})$ has no unobservable modes on the imaginary axis and that both players have a closed-loop perfect state information pattern, they have shown [9, Theorem 4.8′] that this upper value is finite if and only if γ is sufficiently large. Furthermore, in this case, the upper value is equal to the lower value. This common value equals $x_0^T X x_0$ where X is the stabilizing solution of an algebraic Riccati equation. □

The aim here is twofold. Firstly, the aim is to generalize the results of Başar and Bernhard to a framework with a general symmetric matrix Q, i.e. it will be shown (Theorem 7.2.4 and Corollary 7.2.5 below) under an extra assumption that if the relevant algebraic Riccati equation has a stabilizing solution satisfying an additional stability requirement, the value (7.3) can be obtained from a saddle-point solution for the game $(\mathcal{F}, L_2^q(0,\infty), J(\cdot,\cdot,x_0))$. A second aim is to obtain a converse statement, i.e. to show that if the value (7.3) exists, the stabilizing solution of the relevant algebraic Riccati equation also exists (Theorem 7.2.7 below). For the second purpose, it turns out that the set of linear stabilizing state feedback controls needs to be further restricted.

As a preliminary result the following lemma concerning the system (7.1) with the criterion (7.2) is needed.

Lemma 7.2.3 *Let $S := BR^{-1}B^T$, $W := EV^{-1}E^T$. Let X be an arbitrary symmetric $n \times n$ matrix, then*

$$J(F, w, x_0) = x_0^T X x_0 + \int_0^\infty \Big(x^T(Q + A^T X + XA - XSX + XWX)x + \\ + \left|(F + R^{-1}B^T X)x\right|_R^2 - \left|w - V^{-1}E^T Xx\right|_V^2 \Big)\, dt. \tag{7.4}$$

Proof According to Lemma 2.1.2, $x(t) \to 0$ for $t \to \infty$. Thus

$$J(F, w, x_0) = \int_0^\infty \left(x^T(Q + F^T RF)x - w^T V w + \frac{d}{dt} x^T X x - \frac{d}{dt} x^T X x \right) dt =$$

$$= x_0^T X x_0 + \int_0^\infty \Big(x^T(Q + XA + A^T X)x + x^T F^T RFx + 2x^T F^T B^T Xx + \\ -w^T V w + 2w^T E^T Xx \Big)\, dt.$$

Hence, the two completions of the squares

$$x^T F^T RFx + 2x^T F^T B^T Xx = \left|(F + R^{-1}B^T X)x\right|_R^2 - x^T XSXx$$

and

$$w^T V w - 2w^T E^T Xx = \left|w - V^{-1}E^T Xx\right|_V^2 - x^T XWXx,$$

show that (7.4) holds. □

This lemma shows that if X satisfies the algebraic Riccati equation (7.5), an optimal choice for the minimizing player is $-R^{-1}B^T X$, which is an admissible choice if X is the stabilizing solution of this equation. If the maximizing player would be restricted to choose linear state feedback matrices as well, his optimal choice would have been the state feedback matrix $V^{-1}E^T X$. The next theorem shows that under the open-loop information structure, the optimal choice for the maximizing player, given that the minimizing player chooses $-R^{-1}B^T X$, can indeed be obtained from the feedback law $x \mapsto V^{-1}E^T Xx$. This theorem provides a set of sufficient conditions for a saddle-point solution to exist. Consequently, it also generates the upper value of the indefinite differential game, i.e. a solution of Problem 7.2.1.

Theorem 7.2.4 *Consider the linear system* (7.1) *and the quadratic cost functional* (7.2). *Assume that the matrices R and V are positive definite. Let $S := BR^{-1}B^T$, $W := EV^{-1}E^T$. Assume that the algebraic Riccati equation*

$$Q + A^T X + XA - XSX + XWX = 0 \tag{7.5}$$

has a stabilizing solution X. Assume that $A - SX$ is stable. Furthermore, assume that there exists a real symmetric $n \times n$ matrix Y, that satisfies the matrix inequality

$$Q + A^T Y + YA - YSY \geq 0. \tag{7.6}$$

Define

$$\overline{F} := -R^{-1}B^T X, \tag{7.7}$$

$$\overline{w}(t) := e^{t(A-SX+WX)}x_0. \tag{7.8}$$

Then $\overline{F} \in \mathcal{F}$ and $\overline{w} \in L_2^q(0,\infty)$, and the pair $(\overline{F}, \overline{w})$ is a saddle-point solution for the two-person zero-sum game

$$(\mathcal{F}, L_2^q(0,\infty), J(\cdot,\cdot,x_0)). \tag{7.9}$$

The value of this game is equal to $x_0^T X x_0$.

Proof The matrices $A - SX$ and $A - SX + WX$ are stable by assumption, which implies that $\overline{F} \in \mathcal{F}$ and $\overline{w} \in L_2^q(0,\infty)$, respectively. According to Lemma 7.2.3 we have

$$J(F, w, x_0) = x_0^T X x_0 + \int_0^\infty \left(\left|(F - \overline{F})x\right|_R^2 - \left|w - V^{-1}E^T X x\right|_V^2 \right) dt. \tag{7.10}$$

This shows that the feedback matrix $\overline{F}$ satisfies the maximization property of a saddle-point solution. In fact, it follows that

$$J(\overline{F}, w, x_0) = x_0^T X x_0 - \int_0^\infty \left|w - V^{-1}E^T X \tilde{x}\right|_V^2 dt \leq x_0^T X x_0,$$

where $\tilde{x}$ is generated by

$$\dot{\tilde{x}} = (A + B\overline{F})\tilde{x} + Ew, \quad \tilde{x}(0) = x_0.$$

Furthermore, if $J(\overline{F}, w, x_0) = x_0^T X x_0$ then $w = \overline{w}$. Hence

$$J(\overline{F}, w, x_0) < x_0^T X x_0 \text{ for all } w \neq \overline{w}, \text{ and } J(\overline{F}, \overline{w}, x_0) = x_0^T X x_0. \tag{7.11}$$

The next aim is to show that $J(F, \overline{w}, x_0) - J(\overline{F}, \overline{w}, x_0)$ is nonnegative for all $F \in \mathcal{F}$. Let the state variables $\hat{x}$ and $\overline{x}$ be generated by

$$\dot{\hat{x}} = (A + BF)\hat{x} + E\overline{w}, \quad \hat{x}(0) = x_0,$$

$$\dot{\overline{x}} = (A + B\overline{F})\overline{x} + E\overline{w}, \quad \overline{x}(0) = x_0.$$

Define furthermore

$$\nu := (\overline{F} - F)\hat{x},$$
$$\zeta := \overline{w} - V^{-1}E^T X\hat{x}.$$

Then

$$J(F, \overline{w}, x_0) - J(\overline{F}, \overline{w}, x_0) = \int_0^\infty \left(|\nu|_R^2 - |\zeta|_V^2\right) dt.$$

This expression can be rewritten in the following way. Let $\xi := \overline{x} - \hat{x}$. Then

$$\begin{aligned}\dot{\xi} &= (A + B\overline{F})\overline{x} - (A + BF)\hat{x} = (A + B\overline{F})\xi + B(\overline{F} - F)\hat{x} = \\ &= (A + B\overline{F})\xi + B\nu.\end{aligned} \tag{7.12}$$

Furthermore, $\xi(0) = 0$ and $\zeta = V^{-1}E^T X\xi$. Hence

$$J(F, \overline{w}, x_0) - J(\overline{F}, \overline{w}, x_0) = \int_0^\infty \left(\nu^T R\nu - \xi^T XWX\xi\right) dt.$$

Since $\hat{x}, \overline{x} \in L_2^n(0, \infty)$, it follows that $\xi \in L_2^n(0, \infty)$ and $\nu \in L_2^m(0, \infty)$. So $\dot{\xi} \in L_2^n(0, \infty)$ and thus Lemma 2.1.2 implies that $\xi(t) \to 0$ for $t \to \infty$. Together with $\xi(0) = 0$ this shows that

$$\int_0^\infty \frac{d}{dt}\xi^T X\xi dt = 0.$$

Hence

$$\begin{aligned}J(F, \overline{w}, x_0) - J(\overline{F}, \overline{w}, x_0) &= \int_0^\infty \left(\nu^T R\nu - \xi^T XWX\xi - \frac{d}{dt}\xi^T X\xi\right) dt = \\ &= \int_0^\infty \left(\nu^T R\nu - 2\nu B^T X\xi - \xi^T(A^T X + XA - 2XSX + XWX)\xi\right) dt = \\ &= \int_0^\infty \left(\left|\nu - R^{-1}B^T X\xi\right|_R^2 - \xi^T(A^T X + XA - XSX + XWX)\xi\right) dt = \\ &= \int_0^\infty \left(\left|\nu + \overline{F}\xi\right|_R^2 + \xi^T Q\xi\right) dt.\end{aligned}$$

Next, define $\omega := \nu + \overline{F}\xi$. Then (7.12) shows that $\dot{\xi} = A\xi + B\omega$. Since $\xi(0) = 0$ and $\xi(t) \to 0$ for $t \to \infty$ we also have

$$\int_0^\infty \frac{d}{dt}\xi^T Y\xi dt = 0.$$

Hence

$$J(F,\overline{w},x_0) - J(\overline{F},\overline{w},x_0) = \int_0^\infty \left(\omega^T R\omega + \xi^T Q\xi + \frac{d}{dt}\xi^T Y\xi\right) dt =$$

$$= \int_0^\infty \left(\omega^T R\omega + 2\omega^T B^T Y\xi + \xi^T(Q + A^T Y + YA)\xi\right) dt =$$

$$= \int_0^\infty \left(\left|\omega + R^{-1}B^T Y\xi\right|_R^2 + \xi^T(Q + A^T Y + YA - YSY)\xi\right) dt \geq 0,$$

where the final inequality follows by assumption. □

The proof clearly shows that if $Q \geq 0$, the claim of the theorem follows without the condition (7.6). Alternatively, note that if $Q \geq 0$, this condition is in fact trivially satisfied by choosing $Y = 0$. An immediate consequence of Theorem 7.2.4 is the following corollary, which provides a solution for Problem 7.2.1 under the conditions of Theorem 7.2.4.

Corollary 7.2.5 *Consider Problem 7.2.1. Let the assumptions of Theorem 7.2.4 be fulfilled and let X, $\overline{F}$, and $\overline{w}$ be as in this theorem. Then*

$$\min_{F\in\mathcal{F}} \sup_{w\in L_2^q(0,\infty)} J(F,w,x_0) = \max_{w\in L_2^q(0,\infty)} J(\overline{F},w,x_0) = x_0^T X x_0, \tag{7.13}$$

$$\max_{w\in L_2^q(0,\infty)} \inf_{F\in\mathcal{F}} J(F,w,x_0) = \min_{F\in\mathcal{F}} J(F,\overline{w},x_0) = x_0^T X x_0. \tag{7.14}$$

The rest of this section is devoted to the derivation of a converse statement, i.e. showing that if (7.13) holds for some $\overline{F} \in \mathcal{F}$, then the algebraic Riccati equation (7.5) has a solution. It turns out that the set of admissible state feedback matrices needs to be restricted in order to obtain such a result. The following theorem is a preliminary result.

Theorem 7.2.6 *Consider the system*

$$\dot{x} = Ax + Ew, \tag{7.15}$$

with A stable and the cost functional $\varphi : L_2^q(0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$\varphi(w,x_0) = \int_0^\infty \left(x^T Qx - w^T Vw\right) dt, \tag{7.16}$$

with $Q = Q^T$, $V > 0$, and x is generated by (7.15) with initial state x_0. Let $W := EV^{-1}E^T$. The following conditions are equivalent:

(i) For each $x_0 \in I\!R^n$ *there exists a* $\overline{w} \in L_2^q(0,\infty)$ *such that* $\varphi(w, x_0) \leq \varphi(\overline{w}, x_0)$.

(ii) The Hamiltonian matrix

$$H := \begin{bmatrix} A & W \\ -Q & -A^T \end{bmatrix} \tag{7.17}$$

has no eigenvalues on the imaginary axis.

(iii) The algebraic Riccati equation

$$Q + A^T X + XA + XWX = 0 \tag{7.18}$$

has a stabilizing solution.

Assume that these statements hold. Then the maximum of $\varphi(\cdot, x_0)$ *is uniquely attained by*

$$\overline{w}(t) := V^{-1}E^T X e^{t(A+WX)} x_0 \tag{7.19}$$

where X *is the stabilizing solution of* (7.18). *Furthermore* $\varphi(\overline{w}, x_0) = x_0^T X x_0$.

Proof The equivalence between the statements (i), (ii), and (iii) is proven via the cycle (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). The second part of the theorem is contained in the proof of (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): This implication is shown via the maximum principle for infinite horizon problems [30, 108]. Let $x_0 \in I\!R^n$ and $\overline{w}$ be the element of $L_2^q(0,\infty)$ corresponding to statement (i). Denote the state trajectory corresponding to $\overline{w}$ by $\overline{x}$. Then the maximum principle implies that there exists a costate variable p such that

$$\begin{aligned} \dot{\overline{x}} &= A\overline{x} + E\overline{w}, \quad \overline{x}(0) = x_0, \\ \dot{p} &= -Q\overline{x} - A^T p, \\ \overline{w}(t) &= \arg\max_{\omega \in I\!R^q} \left(\overline{x}(t)^T Q \overline{x}(t) - \omega^T V \omega + 2p(t)^T A\overline{x}(t) + 2p(t)^T E\omega \right). \end{aligned}$$

A completion of the squares shows that

$$-\omega^T V\omega + 2p(t)^T E\omega = -(\omega - V^{-1}E^T p(t))^T V(\omega - V^{-1}E^T p(t)) - p(t)^T W p(t).$$

Since $V > 0$, it follows that $\overline{w}(t) = V^{-1}E^T p(t)$. Hence

$$\begin{bmatrix} \dot{\overline{x}} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & W \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} \overline{x} \\ p \end{bmatrix} = H \begin{bmatrix} \overline{x} \\ p \end{bmatrix}, \quad \overline{x}(0) = x_0.$$

Since $\overline{w} \in L_2^q(0,\infty)$ and A is stable, $\overline{x} \in L_2^n(0,\infty)$. Thus $\dot{\overline{x}} \in L_2^n(0,\infty)$ and according to Lemma 2.1.2, this implies that $\overline{x}(t) \to 0$ for $t \to \infty$ for all $x_0 \in I\!R^n$. This shows that the spectral subspace corresponding to the eigenvalues in the open left-half plane has at least dimension n. Since H is an Hamiltonian matrix this implies that H has no eigenvalues on the imaginary axis.

(ii) $\Rightarrow$ (iii): This implication follows from [128, Theorem 13.6], which states that if H has no imaginary eigenvalues and W is either positive semi-definite or negative semi-definite, then (iii) holds if and only if (A, W) is stabilizable. Clearly, here W is positive semi-definite and A is stable implying that (ii) $\Rightarrow$ (iii) immediately follows.

(iii) $\Rightarrow$ (i): Let $w \in L_2^q(0,\infty)$ and $x_0 \in I\!R^n$. Let x be generated by (7.15). Since A is stable, $x \in L_2^n(0,\infty)$. Thus $\dot{x} \in L_2^n(0,\infty)$ and according to Lemma 2.1.2, this implies that $x(t) \to 0$ for all $t \to \infty$. Hence a completion of the squares shows that

$$\varphi(w,x_0) = \int_0^\infty \left(x^TQx - w^TVw + \frac{d}{dt}x^TXx - \frac{d}{dt}x^TXx \right) dt =$$
$$= x_0^TXx_0 - \int_0^\infty \left| w - V^{-1}E^TXx \right|_V^2 dt.$$

Hence $\varphi(w,x_0) \le x_0^TXx_0$ and equality holds if and only if $w = V^{-1}E^Tx$. Substituting this in (7.15) shows that $\varphi(\cdot,x_0)$ is uniquely maximized by $\overline{w}$ as defined by (7.19). □

Motivated by this result the set of admissible state feedback matrices is restricted in the following way. Define for each $F \in \mathcal{F}$ the Hamiltonian matrix

$$H_F := \begin{bmatrix} A+BF & W \\ -Q-F^TRF & -(A+BF)^T \end{bmatrix}. \tag{7.20}$$

Furthermore, define the subset $\overline{\mathcal{F}} \subset \mathcal{F}$ of state feedback matrices by

$$\overline{\mathcal{F}} := \{F \in \mathcal{F} | H_F \text{ has no eigenvalues on the imaginary axis}\}. \tag{7.21}$$

Within this set of feedback matrices a converse statement can be formulated as follows.

Theorem 7.2.7 *Let $S := BR^{-1}B^T$, $W := EV^{-1}E^T$. Assume there exists an $\overline{F} \in \overline{\mathcal{F}}$ such that for each $x_0 \in I\!R^n$ the following holds:*

$$\min_{F\in\overline{\mathcal{F}}} \sup_{w\in L_2^q(0,\infty)} J(F,w,x_0) = \max_{w\in L_2^q(0,\infty)} J(\overline{F},w,x_0). \tag{7.22}$$

Then the algebraic Riccati equation

$$Q + A^TX + XA - XSX + XWX = 0 \tag{7.23}$$

has a stabilizing solution X. Furthermore, the matrix $A - SX$ is stable.

Proof Let $F \in \overline{\mathcal{F}}$. Since H_F has no imaginary eigenvalues on the imaginary axis, the second statement of Theorem 7.2.6 holds with A, Q, and $\varphi(w, x_0)$ replaced by $A + BF$, $Q + F^T RF$, and $J(F, w, x_0)$, respectively. Hence, according to this theorem,

$$\overline{J}(F, x_0) := \max_{w \in L_2^q(0,\infty)} J(F, w, x_0) = x_0^T \psi(F) x_0,$$

where $\psi : \overline{\mathcal{F}} \to I\!R^{n \times n}$ is defined by $\psi : F \mapsto X$ with X the stabilizing solution of the algebraic Riccati equation

$$Q + F^T RF + (A + BF)^T X + X(A + BF) + XWX = 0.$$

Since eigenvalues depend continuously on the corresponding matrix elements [61, Appendix D], the set $\mathcal{F}$ is open. The subset $\overline{\mathcal{F}}$ is open for the same reason. In [76, Section 11.3] it has been shown that the maximal solution of

$$\tilde{X}(\mu)\tilde{D}(\mu)\tilde{X}(\mu) - \tilde{X}(\mu)\tilde{A}(\mu) - \tilde{A}^T(\mu)\tilde{X}(\mu) - \tilde{C}(\mu) = 0 \tag{7.24}$$

is a real-analytic function of k real variables $\mu \in \Omega$, where Ω is an open connected set in $I\!R^k$ if (i) $\tilde{A}(\mu)$, $\tilde{C}(\mu)$, and $\tilde{D}(\mu)$ depend in the same way on μ, (ii) $\tilde{D}(\mu)$ is positive semi-definite, (iii) $(\tilde{A}(\mu), \tilde{D}(\mu))$ is stabilizable, and (iv) the matrix

$$\begin{bmatrix} -\tilde{A}(\mu) & \tilde{D}(\mu) \\ \tilde{C}(\mu) & \tilde{A}(\mu)^T \end{bmatrix}$$

has no eigenvalues on the imaginary axis for all $\mu \in \Omega$. Under the conditions (ii) and (iii), the maximal solution of (7.24) coincides with the unique solution of (7.24) for which $\sigma(\tilde{A}(\mu) - \tilde{D}(\mu)\tilde{X}(\mu))$ lies in the closed left-half plane [76, Theorem 7.9.3]. Note that $-X$ is the maximal solution of (7.24) with $\tilde{A}(\mu) = A + BF$, $\tilde{C}(\mu) = -Q - F^T RF$, $\tilde{D}(\mu) = W$, and $\mu = \operatorname{vec} F$. Clearly, condition (i) and (ii) hold; condition (iii) follows from the stability of $A + BF$, and condition (iv) follows from the easily verifiable fact that the matrices H_F and

$$\begin{bmatrix} -A - BF & W \\ -Q - F^T RF & (A + BF)^T \end{bmatrix}$$

have the same spectrum. Hence, ψ is an analytic function of F in any open connected subset of $\overline{\mathcal{F}}$. In particular $\overline{J}$ is differentiable with respect to F in such a set. In the following a differentiation argument is applied to $\overline{J}$; see Section 2.1 for details concerning matrix differentiation. By assumption, $\overline{J}$ attains its minimum at $\overline{F} \in \overline{\mathcal{F}}$ for each $x_0 \in I\!R^n$. Choose an arbitrary open connected subset Ω of $\overline{\mathcal{F}}$ containing $\overline{F}$. Since $\overline{\mathcal{F}}$ is open, such a choice is possible. Then [81, Section 7.4, Theorem 1] $\delta_1 \overline{J}(\overline{F}, x_0; \Delta F) = 0$ for all increments ΔF and for all $x_0 \in I\!R^n$. Since $\delta_1 \overline{J}(\overline{F}, x_0; \Delta F) = x_0^T \delta\psi(\overline{F}; \Delta F) x_0$, it follows that $\delta\psi(\hat{F}; \Delta F) = 0$ for all increments ΔF, or, equivalently,

$$\partial\psi(\overline{F}) = 0. \tag{7.25}$$

Define the transformation $\Psi : \mathcal{F} \times I\!R^{n\times n} \to I\!R^{n\times n}$ by

$$\Psi(F, X) = Q + F^T RF + (A + BF)^T X + X(A + BF) + XWX.$$

By definition, we have $\Psi(F, \psi(F)) = 0$ for all $F \in \overline{\mathcal{F}}$. Taking the derivative of this equality and applying the chain rule yields

$$\partial_1 \Psi(F, \psi(F)) + \partial_2 \Psi(F, \psi(F)) \partial\psi(F) = 0 \text{ for all } F \in \Omega.$$

By substituting $F = \overline{F}$ and using (7.25) it follows that $\partial_1 \Psi(\overline{F}, \psi(\overline{F})) = 0$, or, equivalently,

$$\delta_1 \Psi(\overline{F}, \psi(\overline{F}); \Delta F) = 0 \text{ for all } \Delta F. \tag{7.26}$$

The differential of Ψ with respect to its first argument with increment ΔF can easily be computed. Indeed,

$$\begin{aligned}
\delta_1 \Psi(F, X; \Delta F) &= \lim_{\alpha \to 0} \frac{\Psi(F + \alpha\Delta F, X) - \Psi(F, X)}{\alpha} = \\
&= \lim_{\alpha \to 0} \frac{\alpha\Delta F^T(RF + B^T X) + \alpha(F^T R + BX)\Delta F + \alpha^2 \Delta F^T R \Delta F}{\alpha} = \\
&= \Delta F^T(B^T X + RF) + (XB + F^T R)\Delta F.
\end{aligned}$$

Consequently, using (7.25),

$$\Delta F^T(B^T \psi(\overline{F}) + R\overline{F}) + (\psi(\overline{F})B + \overline{F}^T R)\Delta F = 0 \text{ for all } \Delta F.$$

Thus, it follows that $B^T\psi(\overline{F}) + R\overline{F} = 0$, or, equivalently, $\overline{F} = -R^{-1}B^T\psi(\overline{F})$. Substituting this in $\Psi(\overline{F}, \psi(\overline{F})) = 0$ yields

$$Q + A^T\psi(\hat{F}) + \psi(\overline{F})A - \psi(\overline{F})S\psi(\overline{F}) + \psi(\overline{F})W\psi(\overline{F}) = 0.$$

This shows that $\psi(\overline{F})$ satisfies the equation (7.23) and furthermore, since it is the stabilizing solution of the equation $\Psi(\overline{F}, X) = 0$ it follows that $A + B\overline{F} + W\psi(\overline{F}) = A - S\psi(\overline{F}) + W\psi(\overline{F})$ is stable. Finally, since $\overline{F} \in \overline{\mathcal{F}}$, the matrix $A - S\psi(\overline{F})$ is stable. □

7.3 Soft-Constrained Nash Equilibria

We apply the theory developed in the previous section to N-player infinite-horizon stable deterministically disturbed linear time-invariant differential games, where the information structure of the players is a feedback pattern and where they are restricted to linear time-invariant strategies. The cost functions are of the soft-constrained type. Thus the differential equation is

$$\dot{x} = \left(A + \sum_{j=1}^{N} B_j F_j\right) x + Ew, \tag{7.27}$$

and the cost function of player i is the functional $J_i : \mathcal{F}_N \times L_2^q(0,\infty) \times I\!R^n \to I\!R$, defined by

$$J_i(F_1, \ldots, F_N, w, x_0) = \int_0^\infty \left(x^T \left(Q_i + \sum_{j=1}^{N} F_j^T R_{ij} F_j\right) x - w^T V_i w\right) dt. \tag{7.28}$$

The matrices Q_i and R_{ij} are without loss of generality assumed to be symmetric and furthermore, $R_{ii} > 0$ and $V_i > 0$ for all $i = 1, \ldots, N$. It is assumed that each player looks for a strategy minimizing his cost function with a worst-case disturbance, i.e. player i aims to minimize the adjusted cost function

$$\overline{J}_i(F_1, \ldots, F_N, x_0) := \sup_{w \in L_2^q(0,\infty)} J_i(F_1, \ldots, F_N, w, x_0). \tag{7.29}$$

For large values of $\|V_i\|$, the negative disturbance term in the cost function of player i has a relatively high weight. Since player i maximizes over all disturbances in his adjusted cost function, worst-case disturbances have a relatively small size, i.e. he only incorporates disturbances with a small size in his adjusted cost function. Hence, large values of $\|V_i\|$ express a risky behavior of player i, and for smaller values of this quantity he is more risk-averse.

Using the adjusted cost functions we define the following equilibrium concept for the differential game under consideration.

Definition 7.3.1 An N-tuple $\overline{F} = (\overline{F}_1, \ldots, \overline{F}_N) \in \mathcal{F}_N$ is called a *soft-constrained feedback Nash equilibrium* if for all i the following inequality holds:

$$\overline{J}_i(\overline{F}, x_0) \leq \overline{J}_i(\overline{F}_{-i}(F), x_0) \tag{7.30}$$

for all initial states x_0 and for all $F \in I\!R^{m_i \times n}$ such that $\overline{F}_{-i}(F) \in \mathcal{F}_N$. □

The following theorem, which is a straightforward generalization of Corollary 7.2.5, in particular of (7.13), provides a set of sufficient conditions to obtain soft-constrained feedback Nash equilibria.

Theorem 7.3.2 *Consider an N-player infinite-horizon stable linear time-invariant differential game, where the information structure of the players is a feedback pattern and where they are restricted to linear time-invariant strategies. Let the cost function of player i be given by* (7.28) *(see Section 2.3 for the terminology and notation). Assume that the matrices Q_i and R_{ij} are symmetric and that $R_{ii} > 0$ and $V_i > 0$ for all $i = 1, \dots, N$. Define $S_i := B_i R_{ii}^{-1} B_i^T$, $S_{ij} := B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T$, and $W_i := E V_i^{-1} E^T$ for all $i, j = 1, \dots, N$ with $i \neq j$. Assume there exist N real symmetric $n \times n$ matrices X_i, and N real symmetric $n \times n$ matrices Y_i, such that*

$$Q_i + A^T X_i + X_i A - \sum_{i \neq j}^{N} (X_i S_j X_j + X_j S_j X_i) - X_i S_i X_i + \sum_{j \neq i}^{N} X_j S_{ij} X_j + X_i W_i X_i = 0; \tag{7.31}$$

$$A - \sum_{j=1}^{N} S_j X_j + W_i X_i \text{ is stable for each } i = 1, \dots, N; \tag{7.32}$$

$$A - \sum_{j=1}^{N} S_j X_j \text{is stable}; \tag{7.33}$$

$$Q_i + A^T Y_i + Y_i A - \sum_{j \neq i}^{N} (Y_i S_j Y_j + Y_j S_j Y_i) - Y_i S_i Y_i + \sum_{j \neq i}^{N} Y_j S_{ij} Y_j \geq 0. \tag{7.34}$$

Then the N-tuple of feedback matrices $(\overline{F}_1, \dots, \overline{F}_N)$ with

$$\overline{F}_i := -R_{ii}^{-1} B_i^T X_i \tag{7.35}$$

is a soft-constrained feedback Nash equilibrium and

$$\overline{J}_i(\overline{F}_1, \dots, \overline{F}_N, x_0) = x_0^T X_i x_0. \tag{7.36}$$

Note that if the matrices Q_i are positive semi-definite, the inequalities (7.34) are trivially satisfied with $Y_i = 0$. Similar sets of algebraic Riccati equations as (7.31) showed up in Theorems 5.3.2 and 6.4.3. The fact that the matrix A is replaced by the matrix $\tilde{A}$ in Theorem 6.4.3 is due to the input-output structure considered in Chapter 6. It has the advantage among other things that some additional structural assumption can be imposed on this matrix in order to exclude trivial equilibria; see Remark 6.4.2. The disturbance term in equation (6.51), i.e. $\gamma_i^{-2} X_i E E^T X_i$, coincides with the disturbance term in equation (7.31) if we set $V_i = \gamma_i^2 I$. However, it clearly has a different interpretation. Indeed, in Theorem 6.4.3, the parameters γ_i are endogenously determined by equation (6.126). Here the matrices V_i are exogenous parameters. This makes the set of sufficient conditions of Theorem 7.3.2 easier, at least structurally. The positive definiteness condition (6.122), i.e. $X_i \geq 0$, only shows up in Theorem 6.4.3. This is due to the positive cost functions considered in Chapter 6 (automatically induced by considering an input-output structure). In contrast to this positivity requirement, the additional stability requirement (7.33) needs to be imposed in the indefinite cases considered in Chapter 5 and in the present chapter. To end this discussion, note that in the disturbed case one ends up with the additional stability requirement (7.34) (or 6.124 in Chapter 6).

In the rest of this section we assume that $R_{ij} = 0$ for all $i \neq j$ and we consider the simplified two-player scalar case. Let $A = a$, $Q_i = q_i$, $S_i = s_i$, and $W_i = w_i$. Recall that the numbers s_i and w_i are nonnegative; assume further that $s_i \neq 0$. First, we consider the set of inequalities (7.34). This system reduces to

$$q_1 + 2ay_1 - 2s_2y_1y_2 - s_1y_1^2 \geq 0; \tag{7.37}$$
$$q_2 + 2ay_2 - 2s_1y_1y_2 - s_2y_2^2 \geq 0. \tag{7.38}$$

Under the coordinate transformation $\xi_i = s_iy_i$, these inequalities describe regions, borders of which are the hyperbolas C_1 and C_2 studied in Section 5.6. In the next proposition we show that the inequality constraints are in fact automatically satisfied in the present case.

Proposition 7.3.3 *For all real numbers a, q_1, q_2, s_1, s_2, there exists a pair (y_1, y_2) satisfying the inequalities* (7.37) *and* (7.38).

Proof One can easily geometrically verify the existence of a point on the red line in Figure 5.1 satisfying the inequalities. This can formally be seen as follows. Substituting

$$y_1 = t, \quad y_2 = (a - s_1t)/s_2$$

in the inequalities (7.37) and (7.38) produces $t^2 \geq -q_1/s_1$ and $(s_1t - a)^2 \geq -q_2s_2$, respectively. Clearly, one can find either a sufficiently small or a sufficiently large value of t, which satisfies both inequalities. $\square$.

Thus in the two-player scalar case with zero cross control matrices a set of sufficient conditions for a soft-constrained feedback Nash equilibrium is given by (7.31)-(7.33). This set reduces in this case to

$$q_1 + 2ax_1 + (w_1 - s_1)x_1^2 - 2s_2x_1x_2 = 0; \tag{7.39}$$

$$q_2 + 2ax_2 - 2s_1x_1x_2 + (w_2 - s_2)x_2^2 = 0; \tag{7.40}$$

$$a + (w_1 - s_1)x_1 - s_2x_2 < 0; \tag{7.41}$$

$$a - s_1x_1 + (w_2 - s_2)x_2 < 0; \tag{7.42}$$

$$a - s_1x_1 - s_2x_2 < 0. \tag{7.43}$$

(7.44)

Under the coordinate transformation $\xi_i = s_ix_i$, and the notations $\sigma_i := s_iq_i$ and $\omega_i := w_i/s_i - 1$, this system changes into

$$\sigma_1 + 2a\xi_1 + \omega_1\xi_1^2 - 2\xi_1\xi_2 = 0; \quad \sigma_2 + 2a\xi_2 - 2\xi_1\xi_2 + \omega_2\xi_2^2 = 0; \tag{7.45}$$

$$a + \omega_1\xi_1 - \xi_2 < 0; \qquad a - \xi_1 + \omega_2\xi_2 < 0; \tag{7.46}$$

$$a - \xi_1 - \xi_2 < 0. \tag{7.47}$$

Note that the situation $\omega_1 = \omega_2 = -1$, which corresponds to $\epsilon = 0$, agrees with the undisturbed case, i.e. with the system (5.36)-(5.38). Apparently, essentially two extra parameters, i.e. ω_1 and ω_2, and also two extra linear inequalities appear in the disturbed case compared to the undisturbed case. The number of soft-constrained equilibria generated by Theorem 7.3.2 equals the number of intersection points of the curves described by (7.45), henceforth denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, which lie in the intersection of halfplanes described by the inequalities (7.46) and (7.47), henceforth denoted by $\mathcal{X}$.

Proposition 7.3.4 *The curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are hyperbolas.*

Proof We only prove this proposition for $\mathcal{C}_1$; the proof for $\mathcal{C}_2$ is similar. Rewrite the first equation of (7.45) as

$$\begin{bmatrix}\xi_1 & \xi_2\end{bmatrix} A_{\mathcal{C}_1} \begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} - \begin{bmatrix}2a & 0\end{bmatrix}\begin{bmatrix}\xi_1 \\ \xi_2\end{bmatrix} - \sigma_1 = 0, \quad A_{\mathcal{C}_1} := \begin{bmatrix}-\omega_1 & 1 \\ 1 & 0\end{bmatrix}.$$

The matrix $A_{\mathcal{C}_1}$ has eigenvalues

$$\lambda_1 := -\frac{\omega_1}{2} - \sqrt{1 + \frac{\omega_1^2}{4}}, \quad \lambda_2 := -\frac{\omega_1}{2} + \sqrt{1 + \frac{\omega_1^2}{4}}.$$

It is easily seen that $\lambda_1 < 0$ and $\lambda_2 > 0$ for all values of ω_1. Thus, since $A_{\mathcal{C}_1}$ has a negative and a positive eigenvalue, the curve $\mathcal{C}_1$ is a hyperbola. □

Unfortunately, one can easily construct examples (take for instance $a = 1$, $\omega_1 = 2$, and $\omega_2 = 1$) such that the constraint set $\mathcal{X}$ is empty. In such cases it can immediately be concluded that no soft-constrained equilibria can be obtained from Theorem 7.3.2. The fact that situations exist in which no equilibria are generated by Theorem 7.3.2, shows once more the importance of studying the necessity of the sufficient conditions listed in this theorem.

7.4 Stochastic Interpretation

The theory of Sections 7.2 and 7.3 can be interpreted stochastically by means of a linear exponential quadratic Gaussian (LEQG) criterion. The aim of the present section is only to show the outlines of this stochastic interpretation and not to discuss the model itself in detail. The one-player problem is as follows.

Problem 7.4.1 Consider a linear noisy system

$$\dot{x} = Ax + Bu + Ew \tag{7.48}$$

with w a stationary white Gaussian noise with zero mean and covariance $\mathrm{E}\left(w(t)w(\tau)^T\right) = \delta(t-\tau)$, and the cost functional $L : \mathcal{F} \to I\!R$, defined by

$$L(F) = \lim_{T\to\infty} \frac{1}{T} \log \mathrm{E} \exp\left(\frac{1}{2\gamma^2}\int_0^T \left(x^TQx + u^TRu\right)dt\right). \tag{7.49}$$

The number γ is positive, the matrices Q and R are positive semi-definite and positive definite, respectively. Determine, if it exists, a feedback matrix $\overline{F} \in \mathcal{F}$ such that $L(\overline{F}) \leq L(F)$ for all $F \in \mathcal{F}$. □

It is well-known that this problem is related to the H_∞ control problem [53, 62, 107]. Runolfsson [107] has shown that a solution of Problem 7.4.1 can be obtained from the stabilizing solution of the algebraic Riccati equation

$$Q + A^TX + XA - XSX + \gamma^{-2}XEE^TX = 0, \quad S := BR^{-1}B^T, \tag{7.50}$$

which also generates the saddle-point solution for the soft-constrained differential game related to the H_∞ control problem [9] for γ sufficiently large. More precisely, if (Q, A) is observable, a positive number γ^* can be defined such that (7.50) has a positive semi-definite stabilizing solution if and only if $\gamma > \gamma^*$. In that case the matrix $A - SX$ is also stable. Runolfsson [107, Example 3.1] proved under an additional controllability assumption that this stabilizing solution also generates the solution for Problem 7.4.1 as stated by the next theorem.

Theorem 7.4.2 *Consider Problem 7.4.1. Assume that (A, B) and (A, E) are controllable and that (Q, A) is observable. Let γ^* be the infimum of all numbers γ for which the algebraic Riccati equation* (7.49) *has a positive semi-definite stabilizing solution X. If $(A - SX, E)$ is controllable, then $\overline{F} := -R^{-1}B^T X$ is a solution of Problem 7.4.1 and $L(\overline{F}) = 1/(2\gamma^2)\,\mathrm{tr}(E^T X E)$.*

The link between this theorem and Theorem 7.2.4 is obvious.

Next, consider an N-player infinite-horizon stable stochastically disturbed linear time-invariant differential game, where the information structure of the players is a feedback pattern and where they are restricted to linear time-invariant strategies. See Problem 7.4.1 for the precise assumptions concerning the noise. The system equation is

$$\dot{x} = \left(A + \sum_{j=1}^{N} B_j F_j \right) x + Ew \tag{7.51}$$

and the cost function of player i is the functional $L_i : \mathcal{F}_N \to I\!R$, defined by

$$L_i(F_1, \ldots, F_N) = \lim_{T \to \infty} \frac{1}{T} \log \mathrm{E} \exp \frac{1}{2\gamma_i^2} \left(\int_0^T x^T \left(Q_i + \sum_{j=1}^{N} F_j^T R_{ij} F_j \right) x dt \right), \tag{7.52}$$

The numbers γ_i are positive, the matrices R_{ij} are symmetric, and the matrices Q_i and R_{ii} are positive semi-definite and positive definite, respectively.

Definition 7.4.3 An N-tuple $\overline{F} = (\overline{F}_1, \ldots, \overline{F}_N) \in \mathcal{F}_N$ is called an *LEQG feedback Nash equilibrium* if for all i the following inequalities hold:

$$L_i(\overline{F}) \leq J_i(\overline{F}_{-i}(F)) \tag{7.53}$$

for all matrices E, and for all $F \in I\!R^{m_i \times n}$ such that $\overline{F}_{-i}(F_i) \in \mathcal{F}_N$. □

Theorem 7.4.2 can straightforwardly be generalized to the N-player case, i.e. we easily obtain the following result.

Proposition 7.4.4 *Define* $S_i := B_i R_{ii}^{-1} B_i^T$ *and* $S_{ij} := B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T$ *for all* $i, j = 1, \ldots, N$ *with* $i \neq j$. *Assume there exist* N *real symmetric* $n \times n$ *matrices satisfying*

$$X_i \geq 0; \tag{7.54}$$

$$Q_i + A^T X_i + X_i A - \sum_{j \neq i}^{N} (X_i S_j X_j + X_j S_j X_i) - X_i S_i X_i + \sum_{j \neq i}^{N} X_j S_{ij} X_j + \\ + \gamma_i^{-2} X_i E E^T X_i = 0; \tag{7.55}$$

$$A_i := A - \sum_{j=1}^{N} S_j X_j + \gamma_i^{-2} E E^T X_i \text{ is stable for } i = 1, \ldots, N; \tag{7.56}$$

$$\overline{A} := A - \sum_{j=1}^{N} S_j X_j \text{ is stable}; \tag{7.57}$$

$$(A_i, B_i), (\overline{A}, E), \text{ and } (A_i, E) \text{ are controllable for } i = 1, \ldots, N; \tag{7.58}$$

$$(Q_i, A_i) \text{ is observable.} \tag{7.59}$$

Then the N*-tuple f feedback matrices* $(\overline{F}_1, \ldots, \overline{F}_N)$ *with*

$$\overline{F}_i := -R_i^{-1} B_i^T X_i$$

is a LEQG feedback Nash equilibrium and

$$L_i(\overline{F}) = (1/2\gamma_i^2) \operatorname{tr} \left(E^T X_i E \right).$$

A link between this proposition and Theorem 7.3.2 is obvious; for a more detailed analysis of the conditions listed in this proposition we refer to the discussion following that theorem.

7.5 Concluding Remarks

Concluding remarks on the indefinite soft-constrained differential game are given in Section 8.2.4. The soft-constrained feedback Nash equilibrium concept and its relation to other robust equilibrium concepts introduced in this thesis are discussed in Section 8.1.4. A preliminary version of this chapter has been presented by J.C. Engwerda at the *Fourteenth International Symposium of Mathematical Theory of Networks and Systems* (Perpignan, France, June 2000).

Chapter 8

Concluding Remarks

8.1 Main Results

8.1.1 Moving Horizon Strategies

In Chapter 3 we have introduced a moving horizon solution concept for N-player infinite-horizon time-invariant differential games. Below we summarize the definition, the main theorems, corollaries, propositions, and figures, as listed in the following table.

		page number			page number
Definition 3.2.1	:	32	Corollary 3.3.2	:	36
Theorem 3.3.1	:	35	Proposition 3.2.2	:	33
Theorem 3.3.7	:	39	Proposition 3.3.3	:	37
Theorem 3.3.9	:	41	Proposition 3.3.4	:	38
Theorem 3.3.11	:	42	Proposition 3.3.5	:	38
Theorem 3.4.4	:	56	Figure 3.2	:	43
Theorem 3.4.5	:	58	Figure 3.5	:	55
Theorem 3.4.8	:	61	Figure 3.6	:	59
			Figure 3.7	:	62

In Definition 3.2.1 the moving horizon solution concept has been defined. Moving horizon strategies have the property that at each point in time t the actions of the players correspond to initial actions of an open-loop Nash equilibrium on the time horizon $[t, t+L]$, for some fixed horizon length L. The time-invariance of the differential game has been exploited to construct time-invariant moving horizon strategies. How these specific moving horizon strategies can be determined has been formulated in Proposition 3.2.2.

The moving horizon concept has been further analyzed in a linear-quadratic setting. In this setting, we have shown in Theorem 3.3.1 that unique moving horizon strategies exist if and only if a certain matrix is invertible. It is well-known [45] that open-loop Nash equilibria for finite-horizon linear-quadratic differential games can be obtained from a system of asymmetric coupled Riccati differential equations, under the condition that this system has a solution on the complete planning horizon. Similarly, under the same condition, where one takes the moving horizon length L as planning horizon, the moving horizon strategies can be obtained from the same system of Riccati equations. This is stated in Corollary 3.3.2. In Theorem 3.3.7, we have shown under certain conditions that the control functions corresponding to the moving horizon strategies constitute in the limit $L \to \infty$ an open-loop Nash equilibrium for the infinite-horizon game. The series of Propositions 3.3.3–3.3.5 can be used to determine the moving horizon solution and to study its stability in more detail. These propositions however require restrictive assumptions on the system data. Nevertheless, it has been argued that it is not unlikely that these assumptions are satisfied in applications. Furthermore, they hold in the scalar case; a special case which one often encounters in economic applications. We studied the scalar moving horizon solution comprehensively in Section 3.3.2. In Theorem 3.3.9 it is shown that the moving horizon strategies exist for each horizon length L. This theorem additionally states that if the open-loop system is stable, the moving horizon closed-loop system is also stable for each horizon length L. Alternatively, if the open-loop system is not stable, a critical value has been determined which has the property that the moving horizon closed-loop system is stable if and only if the moving horizon length is larger than this critical value.

The analytic tractability of the moving horizon concept has been illustrated by a government debt stabilization game [115]. This game is played between a fiscal and a monetary authority. They both have their own objective, but they also have the common objective to stabilize the government debt. Closed-loop analytic formulas for the moving horizon government debt and the corresponding fiscal and monetary strategies have been obtained. The results are summarized in the Theorems 3.4.4, 3.4.5, and 3.4.8 and illustrated by the Figures 3.5–3.7. For instance, it has been shown that the government debt dynamics reaches a steady state value for sufficiently large values of the moving horizon length. It has been assumed that the initial debt exceeds the target debt. If this initial debt excess is sufficiently small, the difference between the steady state debt and the target debt is always larger than this difference. Alternatively, if the difference between the initial debt and the target debt is large enough, a critical value exists, which has the property that the difference between the target debt and the steady state debt is smaller than the difference between the initial debt and the steady state debt if and only if the moving horizon length is larger than this critical value.

In conclusion, the analytic computability of the practical relevant moving horizon feedback strategies has clearly been illustrated by the economic example. It is a promising example for other application fields to study the moving horizon concept; especially when feedback Nash equilibria cannot be determined analytically. Such projects will benefit from further theoretical research.

8.1.2 Noncooperative Disturbance Decoupling

Let us briefly review Chapter 4 in mathematical terms. The decoupling problems investigated in that chapter involve a two-player linear input-output differential game where the information structure of the players is a feedback pattern and where they are restricted to linear strategies. The equations are

$$\dot{x} = (A + B_1F_1 + B_2F_2)x + Ew, \quad z_1 = C_1x, \quad z_2 = C_2x.$$

The closed-loop impulse response from the disturbance w to the output z_i of player i is the matrix function $T_i(t) = C_ie^{t(A+B_1F_1+B_2F_2)}E$. The system is said to be disturbance decoupled for player i if $T_i = 0$. A player who aims to construct a feedback matrix F_1 such that the system is disturbance decoupled for him is said to face the *noncooperative disturbance decoupling problem* (NDDP). In solving this problem, he has to take the strategy of the other player into account. In Section 4.3 we have distinguished between two types of solvability of the NDDP:

(i) The NDDP is said to be *uniformly solvable* for player 1 if

$$\exists\, F_1 \in I\!R^{m_1\times n}\ \forall\, F_2 \in I\!R^{m_2\times n}\ T_1 = 0.$$

In Theorem 4.4.4, page 70, it is stated that this the NDDP is uniformly solvable for player 1 if and only if

$$\operatorname{im}\begin{bmatrix} B_2 & E \end{bmatrix} \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1).$$

(ii) The NDDP is said to be *universally solvable* for player 1 if

$$\forall\, F_2 \in I\!R^{m_2\times n}\ \exists\, F_1 \in I\!R^{m_1\times n}\ T_1 = 0.$$

In Theorem 4.5.4, page 73, it is stated that the NDDP is universally solvable for player 1 if and only if

$$\operatorname{im} E \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1);$$
$$\operatorname{im} B_2 \subset \max \mathcal{V}(A, \operatorname{im} B_1, \ker C_1) + \operatorname{im} B_1.$$

The solutions obtained for the two solvability problems of the NDDP are stated in terms of necessary and sufficient conditions involving controlled-invariant subspaces. Such spaces have been extensively studied in the literature [15, 118]; hence, these conditions are easily verified. Furthermore, decoupling feedback strategies can easily be found using standard techniques from geometric control theory. In these respects, the results of Chapter 4 can immediately be applied. This stands in contrast to the results of other chapters, which often involve complicated systems of coupled Riccati-type equations.

8.1.3 Infinite-Horizon Feedback Nash Equilibria

In Chapter 5 we have defined (Definition 5.3.1, page 82) the concept of a feedback Nash equilibrium for the class of N-player infinite-horizon linear-quadratic time-invariant differential games where the information structure of the players is a feedback pattern and where they are restricted to linear time-invariant strategies. Additionally we have imposed a coupled-constraint condition on the strategy spaces, which states that an N-tuple of strategies internally stabilizes the closed-loop system. Furthermore, the state weighting matrices are not required to be positive semi-definite. For this class of differential games, feedback Nash equilibria are characterized by solutions of a coupled system of symmetric algebraic Riccati equations that satisfy an additional stability requirement, i.e. in Theorem 5.3.2, page 83, it is stated that a feedback Nash equilibrium exists if and only if a solution of this system of Riccati equations exists that has the additional stability requirement. Hence, equilibrium properties (e.g. existence, the number of equilibria, equilibrium costs, numerical algorithms) can be studied by means of the set of algebraic Riccati equations, which is not an easy problem; several authors have been working in this direction [46, 51, 99, 79, 122]. This system of Riccati equations does not seem to have received any attention for indefinite state weighting matrices. In Section 5.6 we investigate this system for the two-player scalar case under the additional assumption that the cross control weighting matrices are zero. It turns out that the equilibria are characterized by the intersection points of two hyperbolas in a certain half plane. These hyperbolas are determined by three scalar parameters. One of these parameters completely determines the center, asymptotes, and principal axes of each of the hyperbolas. These characteristics are listed in Table 5.1, page 96. The positions of the hyperbolas with respect to their center is determined by the other two parameters; see Table 5.2, page 96. A number of examples, each showing a different possible situation, are depicted in the Figures 5.2-5.9, 94 and 95. It is shown that zero, one, two, or three equilibria can occur and it is geometrically argued how this depends on the parameters. The results are summarized in Table 5.3, page 98.

	name	information structure	definition	result
A	risk-sensitive	memoryless perfect state	Def. 6.4.1 page 138	Thm. 6.4.3 page 141
B	soft-constrained	feedback	Def. 7.3.1 page 158	Thm. 7.3.2 page 159
C	stochastic	feedback variance-independent	Def. 5.5.1 page 88	Thm. 5.5.2 page 88
D	LEQG	feedback	Def. 7.4.3 page 163	Prop. 7.4.4 page 164

Table 8.1: Overview of the Robust Equilibrium Concepts

8.1.4 Robust Equilibria

The robust equilibrium concepts that have been introduced in this thesis are listed in Table 8.1. The definition and characteristic properties of the concepts A and B will be briefly reviewed here. This review also included a discussion about the sufficient conditions that have been derived to analyze and determine robust equilibria. Concepts C and D are stochastic interpretations of the feedback Nash equilibrium concept (discussed in Section 8.1.3) and concept B, respectively. These interpretations are straightforwardly based on the well-known connections between LQ and LQG control theory (certainty equivalence principle) and between H_∞ and LEQG control theory [62, 53]. Concepts C and D are not further discussed here.

The concepts A and B fit in the scope of N-player infinite-horizon stable deterministically disturbed linear-quadratic time-invariant differential games where the players are restricted to linear time-invariant strategies (see Table 8.1 for the information structures). For simplicity, consider the case $N = 2$. The differential equation involved is given by

$$\dot{x} = (A + B_1F_1 + B_2F_2)x + Ew, \quad x(0) = x_0.$$

The stability property in such a differential game states that the matrix $A + B_1F_1 + B_2F_2$ is stable, i.e. the pair of feedback matrices (F_1, F_2) is required to be in the set $\mathcal{F}_2$ (clearly, this is a coupled-constraint condition). In addition the pair $(A, [B_1\ B_2])$ is required to be stabilizable, thus ensuring that the set $\mathcal{F}_2$ is nonempty. The disturbance w is assumed to be an element of the space $L_2^q(0, \infty)$. In concept A the players have certain beliefs about the size of the disturbance. Each player has a risk-sensitivity parameter r_i expressing this size, i.e. player i believes that the norm of the disturbance is bounded by the parameter r_i. These parameters are exogenous. In contrast, no constraints on the disturbance are considered in concept B. The quadratic cost functions of the players in both concepts[1] are given by (the distinction between the concepts is indicated by superscripts A and B)

$$J_1^A(F_1, F_2, w, x_0) = \int_0^\infty x^T(Q_1 + F_1^T R_{11} F_1 + F_2^T R_{12} F_2)x dt;$$

$$J_2^A(F_1, F_2, w, x_0) = \int_0^\infty x^T(Q_2 + F_1^T R_{21} F_1 + F_2^T R_{22} F_2)x dt;$$

$$J_1^B(F_1, F_2, w, x_0) = \int_0^\infty (x^T(Q_1 + F_1^T R_{11} F_1 + F_2^T R_{12} F_2)x - w^T V_1 w) dt;$$

$$J_2^B(F_1, F_2, w, x_0) = \int_0^\infty (x^T(Q_2 + F_1^T R_{21} F_1 + F_2^T R_{22} F_2)x - w^T V_2 w) dt.$$

For simplicity reasons, in concept A it is assumed that $Q_i \geq 0$. This restriction has not been imposed in concept B. In both concepts it is assumed that $R_{ii} > 0$. The cost functions in concept B are of the soft-constrained type: each player has a soft-constrained parameter V_i which he uses to weight the disturbance term explicitly in his cost function. It is assumed that $V_i > 0$.

The uncertain frameworks for the concepts A and B are designed in such a way that if the disturbance is not active ($w = 0$), the framework coincides with the framework for which feedback Nash equilibria are defined in Chapter 5. For active disturbances, the robustness of the risk-sensitive and soft-constrained equilibria is expressed by adjusting the cost functions according to worst-case disturbances. The adjusted cost functions for the risk-sensitive and soft-constrained concept are given by

$$\overline{J}_i^A(F_1, F_2, x_0) = \sup_{\|w\| \leq r_i} J_i^A(F_1, F_2, w, x_0);$$

$$\overline{J}_i^B(F_1, F_2, x_0) = \sup_{w \in L_2^q(0,\infty)} J_i^B(F_1, F_2, w, x_0).$$

These adjusted cost functions show the specific characteristics of the parameters r_i and V_i. The parameter r_i expresses the level of risk-sensitivity of player i. If the value of r_i is small, player i has relatively low costs but his adjusted cost function incorporates only disturbances with a small size, i.e. he is not risk-averse. For larger values of r_i, player i is more risk-averse, which results in larger costs. The parameter V_i has a similar characteristic, but in an inversely proportional way.

[1] Actually, an input-output structure has been considered in Section 6.4, the section in which concept A has been introduced. The L_2 norm of the outputs represents the cost functions. It is well-known that this representation can easily be written in the format shown here.

Indeed, for large values of $\|V_i\|$, the negative disturbance term in the cost function of player i has a relatively high weight. Since player i maximizes over all disturbances in his adjusted cost function, worst-case disturbances have a relatively small size, i.e. he only incorporates disturbances with a small size in his adjusted cost function. Hence, large values of $\|V_i\|$ express a risky behavior of player i, and for smaller values of this quantity he is more risk-averse.

Using the adjusted cost functions, the robust equilibrium concepts are naturally defined in the following way. A pair of matrices $(\overline{F}_1, \overline{F}_2) \in \mathcal{F}_2$ is called a risk-sensitive memoryless perfect state Nash equilibrium if

$$J_1^A(\overline{F}_1, \overline{F}_2, x_0) \leq J_1^A(F_1, \overline{F}_2, x_0) \text{ for all matrices } F_1 \text{ such that } (F_1, \overline{F}_2) \in \mathcal{F}_2;$$
$$J_2^A(\overline{F}_1, \overline{F}_2, x_0) \leq J_2^A(\overline{F}_1, F_2, x_0) \text{ for all matrices } F_2 \text{ such that } (\overline{F}_1, F_2) \in \mathcal{F}_2.$$

The initial state x_0 is considered to be a fixed parameter in this definition. In general, a pair of equilibrium matrices depends on the initial state. A pair of matrices $(\overline{F}_1, \overline{F}_2) \in \mathcal{F}_2$ is called a soft-constrained feedback Nash equilibrium if

$$J_1^B(\overline{F}_1, \overline{F}_2, x_0) \leq J_1^B(F_1, \overline{F}_2, x_0) \text{ for all } x_0 \in I\!R^n \text{ and } F_1 \text{ such that } (F_1, \overline{F}_2) \in \mathcal{F}_2;$$
$$J_2^B(\overline{F}_1, \overline{F}_2, x_0) \leq J_2^B(\overline{F}_1, F_2, x_0) \text{ for all } x_0 \in I\!R^n \text{ and } F_2 \text{ such that } (\overline{F}_1, F_2) \in \mathcal{F}_2.$$

Here, the initial state x_0 is not part of the information pattern of the players; accordingly, equilibrium matrices are required to be independent of the initial state.

In the Theorems 6.4.3 and 7.3.2 a set of sufficient conditions is presented for robust equilibria in terms of matrix equations. For the case $N = 2$ and $R_{12} = R_{21} = 0$, these conditions are as follows.

Each sextuple $(X_1, X_2, P_1, P_2, \gamma_1, \gamma_2)$, with (X_1, X_2), and (P_1, P_2) pairs of real symmetric $n \times n$ matrices, and (γ_1, γ_2) a pair of nonzero real numbers, that satisfies system I below (with $S_i := B_i B_i^T$), generates a risk-sensitive memoryless perfect state Nash equilibrium.

$$\mathrm{I}\left\{\begin{array}{l} X_1, X_2 \geq 0; \\ Q_1 + A^T X_1 + X_1 A - X_1 S_2 X_2 - X_2 S_2 X_1 - X_1 S_1 X_1 + \gamma_1^{-2} X_1 E E^T X_1 = 0; \\ Q_2 + A^T X_2 + X_2 A - X_2 S_1 X_1 - X_1 S_1 X_2 - X_2 S_2 X_2 + \gamma_2^{-2} X_2 E E^T X_2 = 0; \\ A_1 := A - S_1 X_1 - S_2 X_2 + \gamma_1^{-2} E E^T X_1 \text{ is stable}; \\ A_2 := A - S_1 X_1 - S_2 X_2 + \gamma_2^{-2} E E^T X_2 \text{ is stable}; \\ A_i^T P_i + P_i A_i = -X_i E E^T X_i, \quad i = 1, 2; \\ \gamma_i^{-4} x_0^T P_i x_0 = r_i^2. \end{array}\right.$$

Each quadruple (X_1, X_2, Y_1, Y_2) of real symmetric $n \times n$ matrices, that satisfies system II below (with $S_i := B_i B_i^T$), generates a soft-constrained feedback Nash equilibrium.

$$
\text{II} \begin{cases}
Q_1 + A^T X_1 + X_1 A - X_1 S_2 X_2 - X_2 S_2 X_1 - X_1 S_1 X_1 + X_1 E V_1^{-1} E^T X_1 = 0; \\
Q_2 + A^T X_2 + X_2 A - X_2 S_1 X_1 - X_1 S_1 X_2 - X_2 S_2 X_2 + X_2 E V_2^{-1} E^T X_2 = 0; \\
A_1 := A - S_1 X_1 - S_2 X_2 + E V_1^{-1} E^T X_1 \text{ is stable;} \\
A_2 := A - S_1 X_1 - S_2 X_2 + E V_2^{-1} E^T X_2 \text{ is stable;} \\
A - S_1 X_1 - S_2 X_2 \text{ is stable;} \\
Q_1 + A^T Y_1 + Y_1 A - Y_1 S_2 Y_2 - Y_2 S_2 Y_1 - Y_1 S_1 Y_1 \geq 0; \\
Q_2 + A^T Y_2 + Y_2 A - Y_2 S_1 Y_1 - Y_1 S_1 Y_2 - Y_2 S_2 Y_2 \geq 0.
\end{cases}
$$

System I has the following interpretation. The first inequality requirements are due to the positivity of the cost functions. The second and the third equation form a system of coupled algebraic Riccati equations, which typically shows up in the context of a Nash equilibrium with feedback or memoryless perfect state information patterns and linear time-invariant strategies. The fourth (fifth) condition in system I states stability of the closed-loop system in a risk-sensitive memoryless perfect state Nash equilibrium and under a worst-case disturbance of player 1 (player 2). The disturbance terms in the second up to fifth condition involve the two real unknowns γ_1 and γ_2, which are endogenously determined. They follow from the sixth and the seventh condition in system I, i.e. they are specified by the risk-sensitivity parameters and the initial state. Actually, the sixth equation is artificial: due to the stability requirement, this Lyapunov equation always has a unique real symmetric solution. The seventh equation states that worst-case disturbances for player i have norm equal to the value of his risk-sensitivity parameter.

The first four conditions of system II have similar interpretations as in system I. The fifth condition states that a soft-constrained feedback Nash equilibrium is internally stabilizing. In system I, such a condition did not show up because it is automatically satisfied due to the positivity requirements of the matrices X_i in that system. The last two conditions in system II are decoupled and show up due to the possible indefiniteness of the matrices Q_1 and Q_2.

If $E = 0$, both systems coincide with the system characterizing feedback Nash equilibria for infinite-horizon games (see Theorem 5.3.2). The complexity of that system has been illustrated by the scalar case in Section 5.6. Clearly systems I and II are even more complex. System I has an extra difficulty in the sense that the parameters γ_i are endogenously determined. This in contrast to system II, in which the corresponding parameters V_i are exogenous. The complexity of systems I and II has been illustrated by an analysis of the scalar case in Sections 6.4 and 7.3, respectively. In these sections, one finds a further elaboration on the specific problems encountered in this simplified case.

Both robust equilibrium concepts which have been introduced in the context of deterministically disturbed differential games, were motivated from the successful development of H_∞ control theory. The complexity of the systems I and II show that generalizations from control problems to nonzero-sum differential games are by no means straightforward. Nevertheless, the results in this thesis, i.e. systems I and II, provide a suitable starting point for investigating robust equilibria in economic applications.

8.2 Other Results

8.2.1 An Equivalence Result in LQ Control Theory

In preparation of characterizing infinite-horizon feedback Nash equilibria in Chapter 5, an equivalence result in the context of the indefinite regular zero-endpoint infinite-horizon linear-quadratic (IRZILQ) control problem [119] has been derived in Section 5.2. Specifically, it has been shown that if the class of admissible control functions is restricted to those which can be written as linear time-invariant state feedback laws, the IRZILQ problem has an initial state-independent solution if and only if the corresponding algebraic Riccati equation has a stabilizing solution. The necessity part of this statement is a substantial contribution to the rich LQ control-theoretical literature. This part has been shown by means of a matrix differentiation argument.

8.2.2 Bounded Worst-Case Disturbances

In a risk-sensitive memoryless perfect state Nash equilibrium, each player has his own worst-case disturbances. These disturbances depend on his risk-sensitivity parameter, i.e. on his upper bound of the disturbance norm. In Section 6.2, we studied existence and uniqueness problems of such bounded worst-case disturbances for a given internally stable closed-loop system and a given upper bound. Specifically, we considered the system

$$\dot{x} = Ax + Ew, \quad x(0) = x_0, \quad z = Cx$$

with A stable. For a given positive number r, bounded worst-case disturbances are elements of $L_2^q(0,\infty)$ that maximize $\|z\|$ subject to the constraint $\|w\| \leq r$. It has been shown that the H_∞ norm γ^* of the system plays an important role in this maximization problem. It is well-known that this number γ^* is equal to the L_2 operator norm of the bounded linear operator $\mathcal{G} : w \mapsto z$ where z is the output of the system for zero initial state. Theorem 6.2.13, page 109, states that for each $\gamma > \gamma^*$ the algebraic Riccati equation

$$C^T C + A^T X + XA + \gamma^{-2} X E E^T X = 0$$

has a stabilizing solution X_γ. Furthermore, if $r < \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$, with

$$w_\gamma := \gamma^{-2} E^T X_\gamma \xi, \quad \dot{\xi} = (A + \gamma^{-2} E E^T X_\gamma)\xi, \quad \xi(0) = x_0$$

then there exists a $\hat{\gamma} > \gamma^*$ such that the disturbance $w_{\hat{\gamma}}$ is a bounded worst-case disturbance. It has been argued that one typically expects $\|w_\gamma\|$ to approach ∞ in the limit $\gamma \downarrow \gamma^*$. Hence, in this case, a bounded worst-case disturbance exists.

Theorem 6.2.13 is not completely satisfactory for the following three reasons: (i) it contains the sufficient condition $r < \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$, (ii) this condition is implicitly formulated with respect to the data of the problem, and (iii) Theorem 6.2.13 does not answer the question whether or not $w_{\hat{\gamma}}$ is a unique bounded worst-case disturbance. The problem has been further analyzed in Section 6.2.2 in order to address these issues. In this section the maximization problem has been put in a more general framework: given two Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, a bounded linear operator T from $\mathcal{X}$ to $\mathcal{Y}$, an element $y_0 \in \mathcal{Y}$, and a positive number $r > 0$, maximize $\|y_0 + Tx\|$ subject to $\|x\| \leq r$. Clearly, the problem defining bounded worst-case disturbances fits in the scope of this general problem by setting $\mathcal{X} = L_2^q(0,\infty)$, $\mathcal{Y} = L_2^p(0,\infty)$, $T = \mathcal{G}$, and $y_0(t) = Ce^{tA}x_0$. A solution of the general problem is presented in Theorem 6.2.18, page 113. This solution has been used in order to study the three issues formulated above in more detail. For instance, under the condition $r < \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$ it was found that the the number $\hat{\gamma}$ is unique if and only if the function $w_0(t) := E^T P e^{tA} x_0$ is not the zero function, where P is the solution of the Lyapunov equation $A^T P + PA = -C^T C$ (Theorem 6.2.22, page 120). Furthermore, if $\lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$ is equal to infinity, $w_{\hat{\gamma}}$ is a unique bounded worst-case disturbance. If this number is finite, bounded worst-case disturbances $\overline{w}$ can possibly be obtained from solutions of the equation $(\gamma^{*2} - \mathcal{G}^*\mathcal{G})\overline{w} = w_0$ with norm equal to r. However, it has been shown by means of an example (Example 6.2.24, page 125) that this equation may have no solution, implying the existence of situations in which bounded worst-case disturbances do not exist.

8.2.3 Worst-Case Disturbance Attenuation with Nonzero Initial State

In preparation of studying risk-sensitive memoryless perfect state Nash equilibria in Section 6.4, the corresponding one-player optimal control problem has been investigated in Section 6.3. This problem consists of minimizing the worst-case output of a linear system by choosing an appropriate internally stabilizing linear feedback law. Here the worst-case output is the output which is maximal with respect to norm-bounded disturbances. Specifically, the system under consideration is given by

$$\dot{x} = (A + BF)x + Ew, \quad x(0) = x_0, \quad z = (C + DF)x$$

and the minmax problem can be formulated as finding a feedback matrix $F \in \mathcal{F}$ that attains the value

$$\min_{F \in \mathcal{F}} \sup_{w \in B_r} \|z\|^2. \tag{8.1}$$

This problem has been approached using zero-sum game theory, i.e. a saddle-point solution of the related two-player zero-sum game $\Sigma = (\mathcal{F}, B_r, \|z\|^2)$ has been determined under certain conditions. These conditions involve the following matrices

$$Q := C^T(I - D(D^TD)^{-1}D^T)C;$$
$$\overline{A} := A - B(D^TD)^{-1}D^TC;$$
$$S := B(D^TD)^{-1}B^T.$$

Theorem 6.3.5, page 133, states that if the algebraic Riccati equation

$$Q + \overline{A}^T X + X\overline{A} - XSX + \gamma^{-2}XEE^TX = 0 \tag{8.2}$$

has a positive semi-definite stabilizing solution X_γ for some $\gamma \neq 0$, and if $\|w_\gamma\| = r$, with

$$w_\gamma := \gamma^{-2}E^TX_\gamma\xi, \quad \dot{\xi} = (A - SX_\gamma + \gamma^{-2}EE^TX_\gamma)\xi, \quad \xi(0) = x_0$$

then the pair (F_γ, w_γ), with

$$F_\gamma := -(D^TD)^{-1}(D^TC + B^TX_\gamma)$$

is a saddle-point solution for the zero-sum game Σ. Hence, under these conditions the matrix F_γ attains the minmax value (8.1). This result has been exploited in Section 6.4 to determine sufficient conditions generating risk-sensitive memoryless perfect state Nash equilibria (system I in Section 8.1.4). The sufficient conditions under which the saddle-point solution (F_γ, w_γ) exists have been further investigated in the part of Section 6.3 following Theorem 6.3.5. It is well-known from H_∞ control theory that if the system under consideration has no zeros on the imaginary axis, then the algebraic Riccati equation (8.2) has a positive semi-definite stabilizing solution if and only if $\gamma > \gamma^*$, where γ^* denotes the infimum of all closed-loop operators $w \mapsto z$ for zero initial state, obtained by varying $F \in \mathcal{F}$. This result has been formulated in Theorem 6.3.7, page 134. The next question that has been investigated is whether or not a $\gamma > \gamma^*$ exists such that $\|w_\gamma\| = r$.

It has been shown that $\|w_\gamma\|$ is decreasing in γ and that it converges to zero in the limit $\gamma \to \infty$. Thus under the condition that $r < \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$, there exists a number $\hat{\gamma} > \gamma$ such that $\|w_{\hat{\gamma}}\| = r$ and the feedback matrix $F_{\hat{\gamma}}$ attains the minmax value (8.1). This result is formulated in Theorem 6.3.10, page 137. It has been argued that one typically expects $\|w_\gamma\|$ to approach ∞ in the limit $\gamma \downarrow \gamma^*$. In this case a saddle-point solution always exists. Although the setting is more complicated here, the condition $r < \lim_{\gamma \downarrow \gamma^*} \|w_\gamma\|$ is of a similar nature as the condition found in Section 6.2 under which bounded worst-case disturbances exist (see also the Section 8.2.2). It is an interesting point for further research to extend results obtained in Section 6.2 to find precise conditions under which the norm $\|w_\gamma\|$ is bounded in the limit $\gamma \downarrow \gamma^*$, and to find conditions under which the number $\hat{\gamma}$ is unique.

8.2.4 Indefinite Soft-Constrained Differential Games

In preparation of studying soft-constrained feedback Nash equilibria in Section 7.3, an indefinite soft-constrained differential game has been investigated in Section 7.2. This game involves a linear system

$$\dot{x} = (A + BF)x + Ew, \quad x(0) = x_0$$

and a quadratic cost functional

$$J(F, w, x_0) = \int_0^\infty \left(x^T(Q + F^T RF)x - w^T V w \right) dt$$

with the matrices R and V positive definite. The matrix Q is not assumed to be positive semi-definite; in particular it can be indefinite. The indefinite soft-constrained differential game is the two-player zero-sum game $(\mathcal{F}, L_2^q(0, \infty), J(\cdot, \cdot, x_0))$. Since the information structure that is considered in the N-player context of Section 7.3 is a feedback pattern, we have been looking for saddle-point solutions (F, w) of the soft-constrained game with the matrix F independent of the initial state. In Theorem 7.2.4, page 150, it is stated that such a saddle-point solution exists if (i) the algebraic Riccati equation

$$Q + A^T X + XA - XBR^{-1}B^T X + XEV^{-1}E^T X = 0 \tag{8.3}$$

has a stabilizing solution X with the additional property that the matrix $A - SX$ is stable, and (ii) if the matrix inequality

$$Q + A^T Y + YA - YBR^{-1}B^T Y \geq 0$$

has a real symmetric solution Y. This latter inequality showed up due to the possible indefiniteness of the matrix Q. Of particular interest for the soft-constrained equilibrium concept is the upper value of the soft-constrained differential game. Clearly this upper value has been straightforwardly derived from the saddle-point solution. A converse statement has been formulated in Theorem 7.2.7, page 155, which states that if the upper value exists, then the algebraic Riccati equation (8.3) has a stabilizing solution with the additional property that the matrix $A - SX$ is stable. This theorem has been shown under a restriction of the set of admissible feedback matrices.

8.3 Open Problems and Further Research

An interesting topic for further research on moving horizon strategies is to obtain more stability results for the class of linear-quadratic games. A local result (Proposition 3.3.4, page 38) has been obtained. However, this result requires the solution of a matrix differential equation and is therefore rather implicit. Easily verifiable conditions are much preferable. For this, one needs to study the dynamics of the eigenvalues of the solution of this differential equation. Proposition 3.3.4 has been derived under some symmetry conditions. It is an important issue to see whether local stability results can be obtained under weaker conditions. Clearly, also global stability results are of interest.

In the two versions of the disturbance decoupling problem studied in Chapter 4, the decoupling player has either no information about the other players' strategy or he has complete information. Problems can be posed in which the decoupling player has the information that the other player is also looking for a decoupling strategy. Similar solvability questions as in Chapter 4 can be addressed to study this version of a noncooperative disturbance decoupling problem.

The nonscalar algebraic Riccati equations which characterize feedback Nash equilibria are a challenging point for further research. Conditions under which one can immediately decide how many solutions these equations have are not known. Likewise, there is a need for numerical algorithms to obtain these solutions. Precise conditions under which the algebraic Riccati equations have zero, one, two, or three solutions in the two-player scalar case can be found along the same lines as for the positive case in [46]. This particular problem is left open in the thesis.

Chapters 6 and 7 contain several open problems. Necessary and sufficient conditions under which $\|w_\gamma\|$ is bounded for $\gamma \downarrow \gamma^*$ in Sections 6.2 and 6.3 were not obtained. Furthermore, the existence of worst-case disturbances in both sections is unclear if the limit $\lim_{\gamma\downarrow\gamma^*} \|w_\gamma\|$ exists and if the disturbance bound is larger than this limit. Numerical solvability of the systems which generate risk-sensitive and soft-constrained Nash equilibria needs more attention. These systems relate to sufficient conditions for robust equilibria. Clearly a set of necessary and sufficient conditions describing all the robust equilibria is preferable; a start in this direction has been made in Chapter 7. Nevertheless, the theory has reached a level in which sufficient material is available to study robust Nash equilibria in economic applications.

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Onzekerheid in differentiaalspelen

Samenvatting

In dit proefschrift is een aantal modellen ontwikkeld waarmee onzekerheid in een differentiaalspel gemodelleerd kan worden. Deze modellen zijn gebaseerd op ontwikkelingen in de systeemtheorie en regeltechniek op het gebied van regelen met voorspellingen op basis van modellen, verstoringsontkoppelend regelen en de H_∞ regeltechniek. Verder is er een kader geformuleerd waarin feedback Nash-evenwichten voor lineair-kwadratische differentiaalspelen met een oneindige beslissingshorizon volledig gekarakteriseerd worden door oplossingen van een stelsel algebraïsche Riccati-vergelijkingen.

In hoofdstuk 3 is een oplossingsconcept gedefinieerd voor de klasse van tijdsinvariante differentiaalspelen gebaseerd op regelen met voorspellingen op basis van modellen. Deze manier van regelen wordt gekenmerkt door het continu opschuiven van de beslissingshorizon met vaste lengte. Het onzekerheidsaspect hangt hier dan ook met name samen met de lengte van de beslissingshorizon in een differentiaalspel. Door het voortdurend opschuiven van een eindige beslissingshorizon zijn de spelers in staat om continu een actie uit te voeren gebaseerd op een verlengde beslissingshorizon. Hierbij is het uiteraard van belang dat de spelers voor hun beslissingen actuele informatie kunnen gebruiken. Om die reden is ervoor gekozen om het oplossingsconcept te definiëren voor differentiaalspelen met een feedback informatiestructuur. De beslissingen die op ieder tijdstip genomen worden zijn echter afkomstig van een open-loop Nash-evenwicht voor het probleem met een eindige beslissingshorizon. Op deze manier zijn open-loop en feedback-elementen samengevoegd in één oplossingsconcept voor differentiaalspelen. Hierdoor is het concept zowel praktisch relevant als analytisch berekenbaar.

Het oplossingsconcept is verder uitgewerkt voor linear-kwadratische spelen, in het bijzonder door een analyse van het scalaire geval en van een economisch voorbeeld op het gebied van de invloed van fiscaal en monetair beleid op de hoogte van de staatsschuld. Deze analyses illustreren de ana-

lytische berekenbaarheid. Verder is er speciaal aandacht besteed aan de stabiliteit van het concept en in het bijzonder aan de relatie tussen stabiliteit en de lengte van de door de tijd schuivende beslissingshorizon. Het blijkt dat een korte beslissingshorizon kan leiden tot instabiele dynamica. Een voldoend lange beslissingshorizon leidt wel tot stabiliteit. In een dergelijk spel behalen de spelers hun voordeel met name in de aanvangsperiode. Op de lange termijn is het concept uiteraard niet voordelig, daar een evenwichtsconcept gebaseerd op een oneindige horizon voor lange-termijn-doelstellingen beter geschikt is.

In de overige hoofdstukken is de onzekerheid in een differentiaalspel gemodelleerd door het aanbrengen van een additieve verstoringsterm in de differentiaalvergelijking. De centrale vraag in hoofdstuk 4 betreft de mogelijkheid voor een speler om een feedback-strategie te kiezen waardoor zijn uitgang ontkoppeld wordt van de verstoring. Uiteraard spelen de acties van de andere spelers hierbij een rol. Er wordt aangenomen dat de andere spelers ook een feedback-strategie kiezen. Op basis van deze informatie is er onderscheid gemaakt tussen twee manieren van oplosbaarheid voor het ontkoppelen van een uitgang van de verstoring. (i) De ontkoppelende speler is niet in staat om de strategieën van de andere spelers waar te nemen; nodige en voldoende voorwaarden op de systeemparameters zijn gevonden waaronder deze speler een ontkoppelende feedback-strategie kan kiezen. (ii) De ontkoppelende speler is wel in staat om de strategieën van de overige spelers waar te nemen; nodige en voldoende voorwaarden zijn gevonden waaronder deze speler voor iedere keuze van feedback-strategieën van de andere spelers een ontkoppelende strategie kan kiezen.

Het feedback-Nash-evenwichtsconcept voor linear-kwadratische spelen met een oneindige beslissingshorizon staat centraal in hoofdstuk 5. Het belangrijkste resultaat van dit hoofdstuk is dat ieder feedback Nash evenwicht in een dergelijk spel correspondeert met een oplossing van het bijbehorende stelsel van algebraïsche Riccati-vergelijkingen die aan een extra stabiliteitseis voldoet. Hierbij is aangenomen dat de spelers niet een strategie kunnen kiezen waardoor de dynamica van het systeem instabiel wordt. Verder hoeven de toestandsgewichten in de kostenfuncties niet noodzakelijk positief-semidefiniet te zijn. Het resultaat is gebaseerd op een resultaat voor een corresponderend regelprobleem: er bestaat een lineaire, begintoestands-onafhankelijke, tijdsinvariante, intern stabiliserende toestandsterugkoppelingswet die een kwadratische kostenfunctie minimaliseert dan en slechts dan als deze gegenereerd wordt door de stabiliserende oplossing van de bijbehorende algebraïsche Riccati-vergelijking. Dit resultaat is een bijdrage aan de linear-kwadratische regeltheorie. Ook wordt er een stochastische interpretatie gegeven van het feedback-Nash-evenwichtsconcept, hetgeen leidt tot een stochastisch variantie-onafhankelijk feedback-Nash-evenwichtsconcept.

Het bestuderen van feedback-Nash-evenwichten in het kader van hoofdstuk 5 reduceert tot het bestuderen van het bijbehorende stelsel van algebraïsche Riccati-vergelijkingen. Belangrijke vragen zijn: hoeveel oplossingen heeft dit stelsel en hoe kunnen deze uitgerekend worden? In het algemeen wordt er aan dit soort problemen minder aandacht besteed in dit proefschrift. Desalniettemin is het twee speler scalaire geval uitgebreid geanalyseerd. Het blijkt dat de oplossingen van de Riccati-vergelijkingen die aan de extra stabiliteitseis voldoen corresponderen met snijpunten van twee hyperbolen in een zeker halfvlak. Met behulp van dit geometrisch inzicht is vastgesteld dat het aantal feedback-Nash-evenwichten kan variëren van nul tot drie. Dit aantal kan eenvoudig met behulp van de systeemparameters en een geometrische analyse bepaald worden.

In de hoofdstukken 6 en 7 is een tweetal robuuste Nash-evenwichtsconcepten gedefinieerd en hun relatie met stelsels Riccati-vergelijkingen bestudeerd. Het kader van hoofdstuk 5 is ook in deze hoofdstukken gebruikt. Dit betekent onder meer dat de spelers niet een strategie kunnnen kiezen waardoor de dynamica intern instabiel wordt. De theorie van de hoofstukken 6 en 7 kan door het overeenkomende kader handig vergeleken worden met hoofdstuk 5. De robuuste evenwichtsconcepten zijn beide geïnspireerd op de H_∞ regeltechniek. De concepten in hoofdstuk 6 en 7 generaliseren respectievelijk het verstoringsverminderingsprobleem met begintoestand ongelijk aan nul en het zwak-begrensde differentiaalspel naar een niet-nulsom differentiaalspelkader.

Het robuuste evenwichtsconcept van hoofstuk 6 heet een risicogevoelig Nash-evenwicht. Iedere speler neemt aan dat de verstoring begrensd is door een gegeven getal. Hij drukt zijn risicogevoeligheid uit door de keuze van dit getal. In het H_∞ regelprobleem wordt ook aangenomen dat de verstoring begrensd is. Dit betreft de berekening van de H_∞ norm. Dit komt namelijk neer op het maximaliseren van het quotient van de normen van de uitgang en verstoring. Hierbij wordt gemaximaliseerd over alle verstoringen in de eenheidsbol. Echter in het criterium wordt de begintoestand gelijk aan nul genomen. Dit is niet het geval in het robuuste evenwichtsconcept van hoofstuk 6. Om de relatie met een stelsel Riccati-vergelijkingen te bestuderen zijn het corresponderende nul- en één-speler geval uitgebreid geanalyseerd. Hieruit is gebleken dat de bij een risiscogevoelig Nash-evenwicht passende informatiestructuur een geheugenloze structuur is met volledige toestandsinformatie.

Het robuuste evenwichtsconcept van hoofdstuk 7 heet een zwak-begrensd Nash-evenwicht. Het verschil met hoofdstuk 6 komt naar voren in drie aspecten: (i) de norm van de verstoring is onbegrensd, (ii) de spelers wegen de verstoring expliciet in hun kostenfunctie met behulp van een negatieve term en (iii) de toestandsgewichten zijn niet noodzakelijk positief-semidefiniet. In een zwak-begrensd Nash-evenwicht minimaliseren de spelers hun maximaal verstoorde kosten. Door de eerste twee aspecten valt de 1-speler situatie samen met het zwak-begrensde differentiaalspel. In dit spel wordt in de literatuur het toestandsgewicht over het algemeen positief-semidefiniet gekozen. Vanwege het derde aspect is daarom het 1-speler geval ook uitgebreid geanalyseerd. Door de eventuele indefinietheid van de toestandsgewichten bestaat het stelsel voldoende voorwaarden dat zwak-begrensde Nash-evenwichten beschrijft niet alleen uit Riccati-vergelijkingen maar ook uit Riccati-ongelijkheden.

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BRAM VAN DEN BROEK received a Master of Science degree in Engineering Mathematics from Eindhoven University of Technology in 1996. His master thesis is entitled "Glass Morphology in Manufacturing Jars". In 1997 he started his Ph.D. research in dynamic game theory at the CentER for Economic Research, Tilburg University. His main fields of interest are applied and numerical mathematics, system and control theory, and applications in physics, economics, and traffic.

Differential game theory can be used to model worst-case design problems or to model situations where several interacting authorities make strategic dynamic decisions. In this thesis a number of differential game models has been introduced which can be used to model uncertainty in a differential game framework. These models are based on developments in system and control theory in the areas of model predictive control, disturbance decoupling control, and H_∞ control theory. Furthermore, a framework has been introduced in which feedback Nash equilibria for infinite-horizon linear quadratic differential games are completely characterized by certain solutions of a system of coupled algebraic Riccati equations.

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